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M a t h e m a t i c s

ANALOGUES OF NIELSEN'S AND MAGNUS'S THEOREMS FOR FREE BURNSIDE GROUPS OF PERIOD 3

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We prove that the free Burnside groups $B(m,3)$ of period 3 and rank $m \geq 1$ have Magnus's property, that is if in $B(m,3)$ the normal closures of r and s coincide, then r is conjugate to s or s^{-1}. We also prove that any automorphism of $B(m,3)$ induced by a Nielsen automorphism of the free group F_m of rank m. We show that the kernel of the natural homomorphism $\mathrm{Aut}(B(2,3)) \to GL_2(\mathbb{Z}_3)$ is the group of inner automorphisms of $B(2,3)$.

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Introduction. In the present paper we prove some theorems on free Burnside groups of period 3. The validity of analogous assertions for absolutely free groups are well known. The starting point is an obvious observation that the automorphism groups of an absolutely free group of rank 1 and a free Burnside group of period 3 and rank 1 are isomorphic (to a cyclic group of order 2).

In 1930 W. Magnus [1] proved the so-called Freiheitssatz and the following theorem: If in a free group F the normal closures of $r \in F$ and $s \in F$ coincide, then r is conjugate to s or s^{-1}. We will say that a group G possesses the *Magnus property*, if for any two elements r, s of G with the same normal closures we have that r is conjugate to s or s^{-1}. In [2–4] it is proved that the fundamental group of any compact surface, except of the nonorientable surface of genus 3, possesses the Magnus property.

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Theorem 1. (An analogue of Magnus's theorem, [1]). A free Burnside group $B(3)$ of any rank has Magnus's property.

Let R_n be a relatively free group with basis $X = \{x_1, \dots, x_n\}$. Any homomorphism from R_n into itself is completely determined by the images of the basis elements. For any $x_i \in X$, let ε_i be the automorphism mapping x_i to x_i^{-1} and leaving other elements of X unchanged. For any different $x_i, x_j \in X$, let λ_{ij} be the automorphism mapping x_i to $x_i x_j$ and leaving other elements of X unchanged. These automorphisms $\varepsilon_i, \lambda_{ij}$ are called Nielsen automorphisms. In 1924 Nielsen (see, for example, [5]) showed that the Nielsen automorphisms generate the full automorphism group $\mathrm{Aut}(F_n)$ of a finitely generated absolutely free group F_n.

Obviously, any automorphism of a free group F of some (finite or infinite) rank induces an automorphism of a relatively free group R of the same rank. Hence, there exists an obvious homomorphism $\tau : \mathrm{Aut}(F) \to \mathrm{Aut}(R)$, and every $\alpha \in \mathrm{Aut}(F)$ induces an $\tau(\alpha) \in \mathrm{Aut}(R)$. Any automorphism from $\tau(\mathrm{Aut}(F))$ is called a *tame* automorphism.

If $\tau(\alpha) = \beta$, then we say that $\beta \in \mathrm{Aut}(R)$ can be *lifted* to $\alpha \in \mathrm{Aut}(F)$. In the general case not every automorphism of a relatively free group is a tame automorphism. As it was proved by Andreadakis [6] and Bachmuth [7], even the free nilpotent groups of finite rank and class 3 have automorphism, which is not induced by an automorphism of free group. On the other hand all automorphisms of a relatively free nilpotent groups of an infinite rank are tame (see [8]).

Let $B(3)$ be a free Burnside group of period 3 and of some rank ≥ 1 and $\mathrm{Aut}_N(B(3))$ be a group of all automorphisms generated by Nielsen automorphisms of $B(3)$. By $\mathrm{Aut}_t(B(3))$ we denote the group of all tame automorphisms of $B(3)$. It is known, that if the rank of $B(3)$ is greater 2, then this is a nilpotent group of class 3. Our next result state that

Theorem 2. (Cf. [5], Prop. 4.1). For any automorphism $\alpha \in \mathrm{Aut}(B(3))$ and for any number p not greater than rank of $B(3)$ there are some free generators $x_{i_1}, x_{i_2}, \dots, x_{i_p}$ of $B(3)$ and an automorphism $\beta \in \mathrm{Aut}_N(B(3))$ such that

$$\alpha(x_{i_1}) = \beta(x_{i_1}), \dots, \alpha(x_{i_p}) = \beta(x_{i_p}).$$

Corollary 1. (An analogue of Nielsen's theorem, 1924). For any finite the rank m the equalities

$$\mathrm{Aut}(B(m,3)) = \mathrm{Aut}_N(B(m,3)) = \mathrm{Aut}_t(B(m,3))$$

hold, that is any automorphism of $B(m,3)$ is a Nielsen (hence a tame) automorphism.

From Nielsen's theorem and from Corollary 1 immediately follows:

Corollary 2. For any finite rank $m > 1$ the homomorphism

$$\tau : \mathrm{Aut}(F_m) \to \mathrm{Aut}(B(m,3)) \text{ is onto.}$$

Comparing the above mentioned result of Bryant and Macedonska from [8] with the Corollary 2 we get

Corollary 3. For any (finite or infinite) rank $m > 1$ the homomorphism

$$\tau : \mathrm{Aut}(F_m) \to \mathrm{Aut}(B(m,3)) \text{ is onto.}$$

Bridson and Voghtman in [9] proved that the group $\mathrm{Aut}(F_m)$ of a free group of rank $m > 1$ is a normal closure of a single transposition (12) (which transposes the generators x_1 and x_2 leaving other generators unchanged). From this and from Corollary 3 follows

Corollary 4. The automorphisms group $\mathrm{Aut}(B(m,3))$ is a normal closure of a single involution (transposition) (12) for any finite rank $m > 1$.

Suppose that $x_i, x_j \in X$ are different free generators. We denote by ρ_{ij} the automorphism of F_m, which maps x_i to $x_j x_i$ and leaves other elements of X unchanged.

The automorphisms λ_{ij}, ρ_{ij} generate a subgroup $\mathrm{Aut}^+(F_m)$ of index 2 (see [10]) in the group $\mathrm{Aut}(F_m)$, where $\mathrm{Aut}^+(F_m)$ is the inverse image of subgroup $SL_m(\mathbb{Z})$ under the homomorphism

$$\mathrm{Aut}(F_m) \to GL_m(\mathbb{Z})$$

induced by the epimorphism $F_m \to \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m}$.

Another classical Nielsen's theorem state that the kernel of the map

$$\mathrm{Aut}(F_m) \to GL_m(\mathbb{Z})$$

is the group of inner automorphisms of F_m (see [5], Prop. 4.5). By analogy, the image of the subgroup $\mathrm{Aut}^+(F_m)$ under the homomorphism

$$\tau_m : \mathrm{Aut}(F_m) \to \mathrm{Aut}(B(m,3))$$

we denote by $\mathrm{Aut}^+(B(m,3))$. By virtue of Corollary 2, the subgroup $\mathrm{Aut}^+(B(m,3))$ also has index 2 in $\mathrm{Aut}(B(m,3))$. The epimorphism $\varepsilon_m : \mathrm{Aut}(B(m,3)) \to GL_m(\mathbb{Z}_3)$ (induced by the epimorphism $B(m,3) \to \underbrace{\mathbb{Z}_3 \oplus \cdots \oplus \mathbb{Z}_3}_{m}$) maps the subgroup $\mathrm{Aut}^+ B(m,3)$ onto $SL_m(\mathbb{Z}_3)$. The following analogue of the above mentioned Nielsen's theorem holds.

Theorem 3. (An analogue of Nielsen's theorem, 1919). The kernel of the natural homomorphism $\varepsilon_2 : \mathrm{Aut}(B(2,3)) \to GL_2(\mathbb{Z}_3)$ is the group of inner automorphisms of $B(2,3)$.

A survey on automorphisms of infinite free Burnside groups $B(m,n)$ and other relatively free groups can be found in [11, 12].

Preliminary Lemmas. By definition the free Burnside group $B(m,n)$ of period n and rank m is the quotient group of the absolutely free group F_m of rank m by a normal subgroup F_m^n generated by elements of the form a^n for all $a \in F_m$.

From the definition it follows that any periodic group of period n with m generators is a quotient of $B(m,n)$. In what follows, the notation $B(3)$ will stand for the Burnside group of period 3 of some > 1 fixed rank with the free generators set $X = \{x_1, x_2, \ldots\}$ (which can be either finite or infinite).

First of all we need the following

Lemma 1. The identities

$$(xy)^3 = 1 \quad \text{and} \quad yxy = x^{-1}y^{-1}x^{-1} \tag{1}$$

are equivalent.

Proof. The proof is obvious. □

L e m m a 2. Any element $y \in B(3)$ permutes with any of its conjugates $x^{-1}yx$.

P r o o f . By definition, from the identity $x^3 = 1$ it follows the equality

$$(xy^{-1})^3(yx^{-1}y^3xy^{-1})(y(x^{-1}y^{-1})^3y^{-1})y^3 = 1.$$

After reducing we get the equality $xy^{-1}xyx^{-1}y^{-1}x^{-1}y = 1$, which means that

$$x \cdot y^{-1}xy = y^{-1}xy \cdot x. \qquad \square$$

L e m m a 3. For any elements $x, y \in B(3)$ and any $k, l \in \mathbb{Z}$ it holds the equality

$$x^k \cdot y^{-1}x^{\,l}y = y^{-1}x^{\,l}y \cdot x^k.$$

P r o o f . Immediately follows from Lemma 2. $\square$

L e m m a 4. If $x = (u_1y^{-1}u_1^{-1})(u_2y^{-1}u_2^{-1})$, then $x = u_3yu_3^{-1} \quad (u_3 \in B(3))$.

P r o o f . Using twice the identity (1), we obtain

$$x = u_1(y^{-1}u_1^{-1}u_2y^{-1})u_2^{-1} = u_1(u_2^{-1}u_1yu_2^{-1}u_1)u_2^{-1} = (u_1u_2^{-1}u_1)y(u_1^{-1}u_2u_1^{-1}) = u_3yu_3^{-1},$$

where $u_3 = u_1u_2^{-1}u_1$. $\square$

L e m m a 5. (Cf. [13, Ch. 18]). For any element $u \in B(3)$ and for any generator $x_i \in X$ one of the following equalities

$$u_1, \tag{2}$$

$$u_1x_iu_2, \tag{3}$$

$$u_1x_i^{-1}u_2, \tag{4}$$

$$u_1x_iu_2x_i^{-1}u_3 \tag{5}$$

holds for some $u_1, u_2, u_3 \in Gp(X \setminus x_i)$.

P r o o f . Let $u = u_1x_i^{\pm 1}u_2x_i^{\pm 1} \dots u_kx_i^{\pm 1}u_{k+1}$, where $u_j \in Gp(X \setminus x_i)$, $j = 1, 2, \dots, k+1$. Then

$$u = (v_1x_i^{\pm 1}v_1^{-1})(v_2x_i^{\pm 1}v_2^{-1}) \dots (v_kx_i^{\pm 1}v_k^{-1})(v_ku_{k+1}) = (\prod_{i=1}^{k} v_ix^{\pm 1}v_i^{-1})(v_ku_{k+1}), \tag{6}$$

where $v_j = u_1u_2 \cdots u_j$, $j = 1, 2, \dots, k$.

By Lemma 3, we can rearange the multipliers in (6) and first write down all the multipliers of the form $v_jx_iv_j^{-1}$, and then write all other multipliers of the form $v_kx_i^{-1}v_k^{-1}$. If in (6) we have two consecutive occurrences of x_i with the same exponent, we use Lemma 4 and reduce the number of x_i's in (6) by one. Then we again rearange the multipliers as mentioned above. Therefore, we can assume that in (6) the exponents of x_i are alternating in sign and $k \le 2$. Hence, we get that every element can be put in one of the form (2)–(5). $\square$

The Proof of Theorem 1. From the equality $\langle\langle x \rangle\rangle = \langle\langle y \rangle\rangle$ it follows that $x \in \langle\langle y \rangle\rangle$. It is easy to verify that then x can be represented in the form

$$x = \prod_{i=1}^{k} u_iy^{\pm 1}u_i^{-1}. \tag{7}$$

By Lemma 3, we have

$$x \cdot y^{-1}x^{\pm 1}y = y^{-1}x^{\pm 1}y \cdot x.$$

So, we can rearange in (7) the multipliers $u_iy^{\pm 1}u_i^{-1}$ and first write down all the multipliers of the form $u_iy^{-1}u_i^{-1}$, and then write down all other multipliers of the form $u_iyu_i^{-1}$. Then, by virtue of Lemma 4, we can assume that $k \le 2$ in (7).

If $k=1$, then $x=u_1y^{-1}u_1^{-1}$ or $x=u_1yu_1^{-1}$ and Theorem 1 is proved.

If $k=2$, then $x=u_1y^{-1}u_1^{-1}u_2yu_2^{-1}$.

In this case we repeatedly apply identity (1) and get

$$x=u_1y^{-1}u_1^{-1}u_2yu_2^{-1}=(yy^{-1})u_1y^{-1}u_1^{-1}u_2yu_2^{-1}=y(y^{-1}u_1y^{-1})u_1^{-1}u_2yu_2^{-1}=$$
$$=yu_1^{-1}yu_1^{-1}u_1^{-1}u_2yu_2^{-1}=yu_1^{-1}(u_2^{-1}u_1^{-1}y^{-1}u_2^{-1}u_1^{-1})u_2^{-1}=$$
$$=yu_1^{-1}u_2^{-1}u_1^{-1}y^{-1}u_1^{-1}u_2^{-1}u_1^{-1}=yu_3y^{-1}u_3^{-1}=[y,u_3],$$

where $u_3=u_1^{-1}u_2^{-1}u_1^{-1}$.

On the other hand from the equality $\langle\langle x\rangle\rangle=\langle\langle y\rangle\rangle$ it follows that $y\in\langle\langle x\rangle\rangle$. Repeating the above arguments, we obtain $y=[x,u_4]$ for some u_4. Since the group $B(3)$ is a class ≤ 3 nilpotent group, we get that $x=1$ according to the equalities

$$x=[y,u_3]=[[x,u_4],u_3]=[[[y,u_3],u_4],u_3]=1.$$

Then, from $\langle\langle x\rangle\rangle=\langle\langle y\rangle\rangle$ we assume that $y=1$. □.

The Proof of Theorem 2. To prove Theorem 2 without loss of generality, we can consider the case of countable rank. Let $X=\{x_1,\dots,x_n,\dots\}$ be a countable set of free generators of the group $B(3)$. Choose a subset

$$Y=\{x_1,x_2,\dots,x_p\}\subset X.$$

Obviously, there exists a subset $\mathbb{Z}=\{x_1,\dots,x_q\}\subset X$ such that $Y\subset Gp(\alpha(\mathbb{Z}))$. We can assume that $p\leqslant q$ and hence $Y\subseteq\mathbb{Z}=\{x_1,\dots,x_q\}$.

Let $U=\alpha(\mathbb{Z})=\{u_1,u_2,\dots,u_q\}$, where $u_i=\alpha(x_i)$.

According to Lemma 5, we consider the generator $x_1\in Y$ and represent elements of U in the form (2)–(5), where $i=1$. Then one of the elements of U needs to have a form (3) or (4). Indeed, if the elements of U have only the form either (2) or (5), then in any word $u\in Gp(U)$ the generator x_1 would have an exponent divisible by three and, hence, x_1 would not belong to $Gp(U)$.

So, we can assume that

$$\alpha(\mathbb{Z})=(u_1,u_2,\dots,u_{i_1-1},u_{k_1}x_1u_{k_2},u_{i_1+1},\dots,u_q),\ \ i\leqslant q.$$

Further, by Nielsen automorphisms of the form λ_{1j}, we find an automorphisms

$$\beta_1'\in\mathrm{Aut}_N(B(3))$$

such that $(\beta_1'(\alpha(x_{i_1})))=u_{k_1}x_1$ and, using Nielsen automorphisms of the form ρ_{1j}, we find an automorphisms $\beta_1'',\in\mathrm{Aut}_N(B(3))$ such that

$$\beta_1''(\beta_1'(\alpha(x_{i_1})))=x_1.$$

Finally, if $i_1\neq 1$, we apply the transposition $(1i_1)$ to the both sides of the equality $\beta_1''(\beta_1'(\alpha(x_{i_1})))=x_1$ and get $(1i_1)(\beta_1''(\beta_1'(\alpha(x_{i_1}))))=(1i_1)(x_1)=x_{i_1}$.

It is well known and it is not difficult to verify that any transposition belongs to the group, generated by Nielsen automorphisms. Denoting $\beta_1=(1i_1)\circ\beta_1''\circ\beta_1'$, we get $\beta_1(\alpha(x_{i_1}))=x_{i_1}$.

Applying this process for the generator x_2 and for the sequence $\beta_1(\alpha(\mathbb{Z}))$, we obtain an automorphism $\beta_2 \in \mathrm{Aut}_N(B(3))$ such that $\beta_2(\beta_1(\alpha(x_{i_2}))) = x_{i_2}$ for some $i_2 \neq i_1$. Note that the construction of β_2 implies that the equality $\beta_2(\beta_1(\alpha(x_{i_1}))) = x_{i_1}$ also holds.

Repeating this process for each generator x_k, $k \leqslant p$, we obtain an automorphism $\beta^{-1} = \beta_p \circ (\beta_{p-1} \circ ...(\beta_2 \circ \beta_1)...) \in \mathrm{Aut}_N(B(3))$ such that $\beta^{-1}(\alpha(x_{i_k})) = x_{i_k}$ for all $k \leqslant p$. Hence, $\alpha(x_{i_k}) = \beta(x_{i_k})$ for all $k = 1,2,\ldots,p$. □

The Proof of Theorem 3. For simplicity we denote by a and b the free generators of $B(2,3)$. Consider an automorphism $\alpha \in \ker(\varepsilon_2)$. Then there exists elements z_1, z_2 from the commutator subgroup $[B(2,3), B(2,3)]$ such that

$$\alpha(a) = az_1, \ \alpha(b) = bz_2.$$

It is well known that the group $B(2,3)$ has 27 elements and

$$[B(2,3), B(2,3)] = C(B(2,3)) = \{1, aba^2b^2, bab^2a^2\}.$$

Hence, $|\mathrm{Inn}(B(2,3)| = \dfrac{|B(2,3)|}{|C(B(2,3))|} = 9$, where $C(B(2,3))$ is the center of $B(2,3)$.

To complete the proof of the assertion, we consider all possible variants for elements $z_1, z_2 \in C(B(2,3)$.

If $\alpha_1(a) = a, \ \alpha_1(b) = b, \ \text{then} \ \alpha_1 = i_e.$

If $\alpha_2(a) = a, \ \alpha_2(b) = b(aba^2b^2) = aba^2, \ \text{then} \ \alpha_2 = i_{a^2}.$

If $\alpha_3(a) = a, \ \alpha_3(b) = b(bab^2a^2) = a^2ba, \ \text{then} \ \alpha_3 = i_a.$

If $\alpha_4(a) = a(aba^2b^2) = b^2ab, \ \alpha_4(b) = b, \ \text{then} \ \alpha_4 = i_b.$

If $\alpha_5(a) = a(aba^2b^2) = b^2ab, \ \alpha_5(b) = b(aba^2b^2) = aba^2, \ \text{then} \ \alpha_5 = i_{ba^2}.$

If $\alpha_6(a) = a(aba^2b^2) = b^2ab, \ \alpha_6(b) = b(bab^2a^2) = a^2ba, \ \text{then} \ \alpha_6 = i_{ab}.$

If $\alpha_7(a) = a(bab^2a^2) = bab^2, \ \alpha_7(b) = b, \ \text{then} \ \alpha_7 = i_{b^2}.$

If $\alpha_8(a) = a(bab^2a^2) = bab^2, \ \alpha_8(b) = b(aba^2b^2) = aba^2, \ \text{then} \ \alpha_8 = i_{a^2b^2}.$

If $\alpha_9(a) = a(bab^2a^2) = bab^2, \ \alpha_9(b) = b(bab^2a^2) = a^2ba, \ \text{then} \ \alpha_9 = i_{ab^2}.$

Therefore, we get $\ker(\varepsilon_2) \subseteq \mathrm{Inn}(B(2,3))$. The converse is obvious and consequently $\ker(\varepsilon_2) = \mathrm{Inn}(B(2,3))$. Theorem 3 is proved.

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