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M a t h e m a t i c s

ON AUTOMORPHISMS OF THE RELATIVELY FREE GROUPS SATISFYING THE IDENTITY $[x^n, y] = 1$

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We prove that if an automorphism φ of the relatively free group of the group variety, defined by the identity relation $[x^n, y] = 1$, acts identically on its center, then φ has either infinite or odd order, where $n \geq 665$ is an arbitrary odd number.

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Introduction. Let m and n be integers such that $m > 1$ and $n \geq 665$ is odd. Consider the set of elementary words $\mathcal{E} = \cup_{\alpha=1}^{\infty} \mathcal{E}_\alpha$ in the group alphabet $\{a_1, a_2, \ldots, a_m\}$ defined in [1] (Chapter VI, §2), see also [2] and an arbitrary countable (or finite) abelian group

$$D = \langle d_1, d_2, \ldots, d_i, \ldots | r = 1, r \in \mathcal{R} \rangle, \tag{1}$$

where $\mathcal{R}$ is some set of words in the group alphabet $d_1, d_2, \ldots, d_i, \ldots$

Denote by $A_D(m,n)$ the group generated by

$$a_1, a_2, \ldots, a_m \quad \text{and} \quad d_1, d_2, \ldots, d_i, \ldots \tag{2}$$

and having the defining relations of the form

$$r = 1 \quad \text{for all} \quad r \in \mathcal{R}, \tag{3}$$

$$a_i d_j = d_j a_i \quad \text{for all} \quad i = 1, 2, \ldots, m \quad \text{and} \quad j \in \mathbb{N}, \tag{4}$$

$$A_j^n = d_j \quad \text{for all} \quad A_j \in \mathcal{E} \quad \text{and} \quad j \in \mathbb{N}. \tag{5}$$

For the case $\mathcal{R} = \{d_i^{-1} d_j | i, j \in \mathbb{N}\}$ it is proved in [3] that the inner automorphism group of $A_D(m,n)$ is a characteristic subgroup in the automorphism group of $A_D(m,n)$.

It is proved by S. Adian and V. Atabekyan [4], that if $m > 1$ is an integer and $n \geq 665$ is an odd integer, then for any abelian group (1) the center of $A_D(m,n)$ is D and for the group

$$C = \langle d_1, d_2, \ldots, d_i, \ldots | [d_k, d_j] = 1, \ k, j \in \mathbb{N} \rangle,$$

the group $A_C(m,n)$ is a free group of rank m in the variety of groups $\mathfrak{C}$ defined by the identity $[x^n, y] = 1$. Moreover, the following statement holds.

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Lemma 1. (see Theorem 3 in [4]). The group $A_C(m,n)$ is torsion-free.

Note that the above mentioned results about the group the groups $A_C(m,n)$ for odd $n > 10^{10}$ using another technic were proved by Ol'shanskii and Ashmanov in [5] (see also [6]).

We will use the following well known Theorem of Bear.

Lemma 2. [7]. Let G be a group and the index of the center of G be finite. Then $[G,G]$ is also finite.

The Main Theorem.

Theorem. If an automorphism φ of the group $A_C(m,n)$ is an involution, then there exists an element $c \in C$, for which $\varphi(c) \neq c$.

Proof. Suppose that φ is an involution of $A_C(m,n)$ and it acts identically on the center. We want to get a contradiction.

Consider the following subgroup

$$K = A_C(m,n) \rtimes \langle \varphi \rangle,$$

of the holomorph $Hol(A_C(m,n))$ of $A_C(m,n)$. Recall that in the holomorph the multiplication is defined by the rule:

$$(a_1,\varphi^t)\cdot(a_2,\varphi^s) = (a_1\varphi^t(a_2),\varphi^{t+s}).$$

If $d \in Z(A_C(m,n)) = C$, then

$$(d,1)\cdot(a,\varphi^t) = (da,\varphi^t) = (ad,\varphi^t) = (a,\varphi^t)\cdot(d,1),$$

because $\varphi^t(d) = d$ for $d \in C$ by the condition of Theorem. Therefore, we get that $(d,1) \in Z(K)$.

Consider an element $h = (a,\varphi^t) \in H$. We have $h^2 = (a_1,1)$, where $a_1 \in A_C(m,n)$, and hence $h^{2n} = (a_1^n,1) \in Z(K)$, since $a_1^n \in Z(A_C(m,n))$ (because the identity $[x^n,y] = 1$ holds in $A_C(m,n)$).

This means that the quotient group $K/Z(K)$ is a periodic group.

The element $(1,\varphi)$ has order 2 in K. Since the automorphism φ is an involution, there is an element $x \in A_C(m,n)$ such that $\varphi(x) \neq x$.

Consider the element $y = (\varphi^{-1}(x^{-1})x,\varphi^{-1})$ in K. One can easily check that $y^2 = (\varphi^{-1}(x^{-1})x\varphi^{-2}(x^{-1})\varphi^{-1}(x),\varphi^{-2}) = (1,1)$. Moreover, the element $(1,\varphi)\cdot y = (x^{-1}\varphi(x),1)$ belongs to the subgroup $A_C(m,n)$ of K. Since $A_C(m,n)/C$ is isomorphic to the free Burnside group $B(m,n)$, $(1,\varphi)\cdot y$ has finite period n in the quotient group $K/Z(K)$.

Consider the subgroup L generated by the subgroup $Z(K)$ and elements y and $(1,\varphi)$ of the group K. It is easy to check that the index of the subgroup $Z(K)$ in L is not greater than $2n$. By Lemma 2, we get that the commutator $[L,L]$ is finite.

On the other hand, since $A_C(m,n)$ is torsion-free (see Lemma 1), any nontrivial commutator in L will have an infinite order. This is a contradiction, which completes the Proof of the Theorem. □

Corollary. If an automorphism φ of the group $A_C(m,n)$ is identity on the center C, then the order of φ is either infinite or odd.

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