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Mathematics

OPERATORS ON THE BESOV SPACES OF HOLOMORPHIC FUNCTIONS ON THE UNIT BALL IN $\mathbb{C}^n$

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In the present paper we consider the Toeplitz-$T^\alpha_{\bar{h}}$ and differentiation-D^δ operators on the Besov spaces $B_p(\beta)$ for all $0 < p < \infty$. We show that $T^\alpha_{\bar{h}} : B_p(\beta) \to B_p(\beta)$ for $\bar{h} \in H^\infty(B^n)$ and $D^\delta : B_p(\beta) \to B_p(\widetilde{\beta})$, where $\widetilde{\beta} = \beta + p\delta$.

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Introduction and Basic Constructions. Let denote the complex Euclidean space of a dimension n by $\mathbb{C}^n$. For any points $z = (z_1, \dots, z_n)$, $\zeta = (\zeta_1, \dots, \zeta_n)$ in $\mathbb{C}^n$, we define the inner product by $\langle z, \zeta \rangle = z_1\overline{\zeta}_1 + \dots + z_n\overline{\zeta}_n$ and note that $|z|^2 = |z_1|^2 + \dots + |z_n|^2$. We denote the open unit ball and its boundary, i.e. the unit sphere, in $\mathbb{C}^n$ by $B^n = \{z \in \mathbb{C}^n, \ \ |z| < 1\}$ and $S^n = \{z \in \mathbb{C}^n \ \ |z| = 1\}$. Further, we denote by $H(B^n)$ the set of holomorphic functions on B^n and by $H^\infty(B^n)$ the set of bounded holomorphic functions on B^n.

If $f \in H(B^n)$, then $f(z) = \sum\limits_m a_m z^m \ (z \in B^n)$, where the sum is taken over all multiindices $m = (m_1, \dots, m_n)$ with nonnegative integer components m_k and $z^m = z_1^{m_1} \dots z_n^{m_n}$. Assuming that $|m| = m_1 + \dots + m_n$ and putting $f_k(z) = \sum\limits_{|m|=k} a_m z^m$ for any $k \geq 0$, one can rewrite the Taylor expansion of f by

$$f(z) = \sum_{k=0}^{\infty} f_k(z), \tag{1}$$

which is called homogeneous expansion of f, since each f_k is a homogeneous polynomial of degree k.

Further, for a holomorphic function f and for a real parameter δ we define the operator of differentiation D^α as follows:

$$D^\delta f(z) = \sum_{k=0}^{\infty} (k+1)^\delta f_k(z).$$

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In particular, if $\delta = 1$, then we write $D^1 f(z) = Df(z)$.

The following properties of D are evident

$$|D^{\delta}(1-\langle z,\zeta\rangle)^{-\alpha}| \le C|1-\langle z,\zeta\rangle|^{-\alpha-1-\delta}, \tag{2}$$

$$D^{\delta}Df(z) = D^{\delta+1}Df(z)C|1-\langle z,\zeta\rangle|^{-\alpha-1-\delta}, \tag{3}$$

$$f(z) = \int_0^1 Df(\langle r,z\rangle)dr. \tag{4}$$

Throughout the paper the capital letters $C(...)$ and C_k stand for different positive constants depending only on the parameters indicated.

We define the holomorphic Besov spaces on the unit ball as follows [1].

Definition. Let $\beta + p - n > 0,\ \ 1 \le p < \infty$. A function $f \in H(B^n)$ is said to be of $B_p(\beta)$, if

$$M_f^p(\beta) = \int\limits_{B^n} \frac{|Df(z)|^p d\nu(z)}{(1-|z|^2)^{n+1-\beta-p}} < +\infty,$$

where $d\nu$ is the volume measure on B^n, normalized so that $\nu(B^n) = 1$.

We introduce the norm in $B_p(\beta)$ by $\|f\|_{B_p(\omega)} = M_f(\omega)$ (($|f(0)|$ is not to be added, since $Df = 0$ implies $f = 0$ for a holomorphic function f). Besides, it is easy to check, that if $p > 1$, $n = 1$ and $\beta = 0$, then $B_p(\beta)$ becomes the classical Besov space [2,3].

In particular, for $p = +\infty$ we shall write $B_\infty(\beta) = B_\beta$, where B_β denotes the β-weighted Bloch space on the ball [4].

In [5–7], one can see some other definitions and some characterizations of holomorphic Besov spaces on B^n.

Let $1 \le p < \infty$ and $f \in B_p(\beta)$. Further, let $\beta + p - n > 0$. Then the function $Df(z)$ has the representation

$$Df(z) = C(n,\beta)\int\limits_{B^n} \frac{(1-|\zeta|^2)^{\beta}Df(\zeta)}{(1-\langle z,\zeta\rangle)^{n+1+\beta}}d\nu(\zeta), \quad z \in B^n. \tag{5}$$

This is a simple consequence from the one-dimensional case (for details, see [8,9]). The following auxiliary lemmas will be used.

Lemma 1. Let $f \in B_p(\beta)$, $0 < p < \infty$. Then

$$|Df(z)| \le \frac{\|f\|_{B_p(\beta)}}{(1-|z|^2)^{1+\beta/p}},\ \ z \in B^n.$$

See the proof in [1], Lemma 2.

Lemma 2. If $1 \le p < \infty$ and $f \in B_p(\beta)$, then

$$|f(z)| \le C\int\limits_{B^n} \frac{(1-|\zeta|^2)^{\beta}|Df(\zeta)|}{|1-\langle z,\zeta\rangle|^{n+\beta}}d\nu(\zeta)$$

for $\beta > n - p$.

Proof. Obviously, by (4) and (5) we get

$$f(z)=C(n,\beta)\int_0^1\int_{B^n}\frac{(1-|\zeta|^2)^\beta Df(\zeta)}{(1-r\langle z,\zeta\rangle)^{n+1+\beta}}d\nu(\zeta)dr=$$

$$=C(n,\alpha)\int_{B^n}(1-|\zeta|^2)^\beta Df(\zeta)\int_0^1\frac{drd\nu(\zeta)}{(1-r\langle z,\zeta\rangle)^{n+1+\beta}}=$$

$$=C(n,\beta)\int_{B^n}\frac{(1-|\zeta|^2)^\beta((1-\langle z,\zeta\rangle)^{n+\beta}-1)}{\langle z,\zeta\rangle(1-\langle z,\zeta\rangle)^{n+\beta}}Df(\zeta)d\nu(\zeta).$$

So we have

$$f(z)=C(n,\beta)\int_{B^n}\frac{(1-|\zeta|^2)^\beta g(\langle z,\zeta\rangle)}{(1-\langle z,\zeta\rangle)^{n+\beta}}Df(\zeta)d\nu(\zeta). \tag{6}$$

It is clear that $g(\langle z,\xi\rangle)=(1-\langle z,\xi\rangle)^{n+m+1}-1)/\langle z,\xi\rangle$ is bounded in B^n. Hence, the desired statement follows. □

Lemma 3. Let $f\in B_p(\beta)$ for some $0<p\le 1$. Then

$$\left(\int_{B^n}|Df(z)|\frac{(1-|z|^2)^{\beta/p}}{(1-|z|^2)^n}d\nu(z)\right)^p\le\int_{B^n}|Df(z)|^p\frac{(1-|z|^2)^{p+\beta}}{(1-|z|^2)^{n+1}}d\nu(z).$$

Proof. We have $|Df(z)|=|Df(z)|^p|Df(z)|^{1-p}$. By Lemma 1, we get

$$|Df(z)|\le|Df(z)|^p\frac{\|f\|^{1-p}_{B^p(\beta)}}{(1-|z|^2)^{(1-p)\beta/p+1}}.$$

Therefore,

$$|Df(z)|\frac{(1-|z|^2)^{\beta/p}}{(1-|z|^2)^n}\le|Df(z)|^p\|f\|^{1-p}_{B^p(\beta)}\frac{(1-|z|^2)^{\beta+p}}{(1-|z|^2)^{n+1}}$$

and, by integration over B^n, we get

$$\int_{B^n}|Df(z)|\frac{(1-|z|)^{\beta/p}}{(1-|z|)^n}d\nu(z)\le\|f\|^{1-p}_{B^p(\beta)}\int_{B^n}|Df(z)|^p\frac{(1-|z|)^{\beta+p}}{(1-|z|)^{n+1}}d\nu(z).$$ □

Lemma 4. [9]. Let $\gamma>-1$ and $\beta-\gamma>0$. Then

$$\int_{B^n}\frac{(1-|\zeta|^2)^\gamma d\nu(\zeta)}{|1-\langle z,w\rangle|^{\beta+n+1}}\le\frac{C}{(1-|z|^2)^{\beta-\gamma}}.$$

It is well known that there exists a function $K:B^n\times B^n\to C$ analytic with respect to the first entry and conjugate analytic with respect to the second entry, such that for every $f\in L_2(B^n)$ and every $w\in B^n$ we have $Pf(w)=\int_{B^n}f(z)K(z,w)d\nu(z)$, where $K(z,w)=(1-\langle z,w\rangle)^{-n-l}$. If $\bar{h}\in L^\infty(B^n)$ we can construct the so called Toeplitz operator $T_{\bar{h}}$, where $T_{\bar{h}}=PM_{\bar{h}}$ by definition. The symbol $M_{\bar{h}}$ stands for the standard multiplication operator. Therefore, we can write

$$T^\alpha_{\bar{h}}(f)(z)=\int_{B^n}\frac{(1-|\xi|^2)^\alpha f(\xi)\overline{h(\xi)}d(\nu)}{(1-\langle z,\xi\rangle)^{n+\alpha+1}}. \tag{7}$$

In the present paper we consider the integral operator of the form (7) and prove the boundedness of this operator on $B_p(\beta)$ for all $0 \leq p < \infty$. Next we show that $D^\delta : B_p(\beta) \to B_p(\widetilde{\beta})$, where $\widetilde{\beta} = \beta + p\delta$.

Main Results.

Theorem 1. Let $0 < p < \infty$, $\bar{h} \in H^\infty(B^n)$, then $T_{\bar{h}}^\alpha : B_p(\beta) \to B_p(\beta)$.

Proof. Let $p \geq 1$ and $\bar{h} \in H^\infty(B^n)$. We show that $T_{\bar{h}}^\alpha(f) \in B_p(\omega)$ for any $f \in B_p(\omega)$. By Lemma 2, we have

$$|f(\xi)| \leq C \int\limits_{B^n} \frac{(1-|t|^2)^\beta |Df(t)|}{|1-\xi, t\rangle|^{\beta+n}} d\nu(t).$$

a) Let $p > 1$. By Hölders inequality and by Lemma 4, we have

$$|f(\xi)|^p \leq \frac{C}{(1-|\xi|^2)^{(\beta-1)p/q}} \int\limits_{B^n} \frac{(1-|t|^2)^{\beta p} |Df(t)|^p}{|1-\langle \xi, t\rangle|^{\beta+n}} d\nu(t).$$

Then we get

$$\begin{aligned} |DT_{\bar{h}}^\alpha f(z)|^p &\leq \left(\int\limits_{B^n} \frac{(1-|\xi|^2)^\alpha |f(\xi)| \cdot |\overline{h(\xi)}| d\nu(\xi)}{|1-\langle z, \xi\rangle|^{n+2+\alpha}} \right)^p \leq \\ &\leq \frac{||h||_\infty}{(1-|z|^2)^{p/q}} \int\limits_{B^n} \frac{(1-|\xi|^2)^\alpha |f(\xi)|^p d\nu(\xi)}{|1-\langle z, \xi\rangle|^{n+2+\alpha}} \end{aligned}$$

and

$$I \equiv \int\limits_{B^n} \frac{|DT_{\bar{h}}^\alpha f(z)|^p d\nu(z)}{(1-|z|^2)^{n+1-p-\beta}} \leq \int\limits_{B^n} (1-|t|^2)^{\beta p} |Df(t)|^p \times$$

$$\times \int\limits_{B^n} \frac{(1-|\xi|^2)^{\alpha-(\beta-1)p/q}}{|1-\langle \xi, t\rangle|^{\beta+n}} \int\limits_{B^n} \frac{(1-|z|^2)^{\beta-n} d\nu(z) d\nu(\xi) d\nu(t)}{|1-\langle z, \xi\rangle|^{n+2+\alpha}}.$$

Using Lemma 4, we then obtain

$$I \leq \int\limits_{B^n} (1-|t|^2)^{\beta p} |Df(t)|^p \int\limits_{B^n} \frac{(1-|\xi|^2)^{\beta-n-1-(\beta-1)p/q} d\nu(z) d\nu(\xi)}{|1-\langle \xi, t\rangle|^{n+\beta}} =$$

$$= ||\bar{h}||_\infty \int\limits_{B^n} \frac{|Df(\xi)|^p d\nu(\xi)}{(1-|\xi|^2)^{n+1-p-\beta}} \leq ||f||_{B_p(\beta)}^p ||\bar{h}||_\infty.$$

b) Let now $p = 1$. We have

$$\int\limits_{B^n} |DT_{\bar{h}}^\alpha f(z)| \frac{d\nu(z)}{(1-|z|^2)^{n-\beta}} \leq ||\bar{h}||_\infty \int_{B^n} \frac{(1-|w|^2)^\beta |Df(w)|}{|1-\langle \xi, w\rangle|^{\beta+n}} \int\limits_{B^n} \frac{(1-|\xi|^2)^\alpha}{|1-\langle z, w\rangle|^{n+\beta+2}} \times$$

$$\times \int\limits_{B^n} \frac{d\nu(z) d\nu(\xi) d\nu(w)}{|1-|z|^2)^{n-\beta}} =$$

$$= ||\bar{h}||_\infty \int\limits_{B^n} (1-|w|^2)^\beta |Df(w)| \int\limits_{B^n} \frac{(1-|\xi|^2)^\alpha}{|1-\langle \xi, w\rangle|^{n+\beta}} \int\limits_{B^n} \frac{d\nu(z) d\nu(\xi) d\nu(w)}{(1-|z|^2)^{n-\beta} |1-\langle z, \xi\rangle|^{n+2+\alpha}} \leq$$

$$\leq ||\bar{h}||_\infty \int\limits_{B^n} (1-|w|^2)^\beta |Df(w)| \int\limits_{B^n} \frac{d\nu(\xi)d\vartheta(w)}{|1-\langle\xi,w\rangle|^{\beta+n}(1-|\xi|^2)^{n+1-\beta}} \leq$$

$$\leq ||\bar{h}||_\infty \int\limits_{B^n} |Df(w)| \frac{(1-|w|^2)^\beta d\nu(w)}{|1-|w|^2)^n} = ||h||_\infty \int_{B^n} \frac{|Df(w)|d\nu(w)}{(1-|w|^2)^{n-\beta}} = ||f||_{B_p(\beta)}||\bar{h}||_\infty.$$

c) Let $0<p<1$ and $\bar{h}\in H^\infty(B^n)$. We show that $T^\alpha_{\bar{h}}: B_p(\beta)\to B_p(\beta^*)$, where $\beta^*=-p\alpha+\alpha+1-p+m-mp$ and $m>-n/p$. Using the properties of $B_p(\beta)$ (see [1]) and Lemma 3, we get

$$|f(\xi)|^p \leq \int_{B^n} \frac{(1-|t|^2)^{(m+n+1)p-n-1}|Df(t)|^p}{|1-\langle\xi,t\rangle|^{(m+n)p}} d\nu(t)$$

for sufficiently large m and

$$|DT^\alpha_{\bar{h}} f(z)|^p \leq ||\bar{h}||^p_\infty \int\limits_{B^n} \frac{(1-|\xi|^2)^{p(\alpha+1+n)-n-1}|f(\xi)|^p d\nu(\xi)}{|1-\langle z,\xi\rangle|^{(n+2+\alpha)p}}.$$

By Lemma 4, we obtain

$$\int\limits_{B^n} \frac{|DT^\alpha_{\bar{h}} f(z)|^p d\nu(z)}{(1-|z|^2)^{n+1-p-\beta}} \leq ||h||^p_\infty \int\limits_{B^n} \frac{1}{(1-|z|^2)^{n+1-p-\beta}} \times$$

$$\times \int\limits_{B^n} \frac{(1-|\xi|^2)^{p(\alpha+n+1)-n-1}}{|1-\langle z,\xi\rangle|^{(n+2+\alpha)p}} \int\limits_{B^n} \frac{(1-|t|^2)^{(\beta+n+1)p-n-1}|Df(t)|^p}{|1-\langle\xi,t\rangle|^{(\beta+n)p}} d\nu(t)d\nu(z)d\nu(\xi) =$$

$$= ||\bar{h}||^p_\infty \int\limits_{B^n} (1-|t|^2)^{(m+n+1)p-n-1}|Df(t)|^p \int\limits_{B^n} \frac{(1-|\xi|^2)^{p(\alpha+n+1)-n-1}}{|1-\langle\xi,t\rangle|^{(n+m)p}} \times$$

$$\times \int\limits_{B^n} \frac{(1-|z|^2)^{\beta+p-n-1}}{|1-\langle z,\xi\rangle|^{(n+2+\alpha)p}} d\nu(z)d\nu(t)d\nu(\xi) \leq$$

$$\leq ||\bar{h}||^p_\infty \int\limits_{B^n} (1-|t|^2)^{(m+n+1)p-n-1}|Df(t)|^p \int\limits_{B^n} \frac{(1-|\xi|^2)^{p(\beta-n-1}}{|1-\langle\xi,t\rangle|^{(n+m)p}} \leq$$

$$\leq ||\bar{h}||^p_\infty \int\limits_{B^n} \frac{(1-|t|^2)^{(m+n+1)p-n-1}|Df(t)|^p d\nu(t)}{(1-|t|)^{(m+n)p-\beta}} =$$

$$= ||\bar{h}||_\infty \int\limits_{B^n} \frac{|Df(\xi)|^p d\vartheta(\xi)}{|1-|\xi|^2)^{n+1-p-\beta}} \leq ||f||^p_{B_p(\omega)}||\bar{h}||_\infty. \qquad \square$$

Corollary. Let $f\in B_p(\beta)$, $\|f\|_{B_p(\beta)}\leq 1$, $\bar{h}\in H^\infty$, $0<p<\infty$. Then there exists a constant C independent of f and $\bar{h}$ such that

$$\|T_{\bar{h}}\| \leq C\|\bar{h}\|_{H^\infty}. \tag{8}$$

Proof. According to the proof of Theorem 1, we have: if $f\in B_p(\beta)$, $\bar{h}\in H^\infty$, $0<p<\infty$, then $T_{\bar{h}}(f)\in B_p(\beta)$ and $\|T_{\bar{h}}(f)\|_{B_p(\beta)}\leq C\|f\|_{B_p(\beta)}\|\bar{h}\|_\infty$. $\square$

The next theorem is about the boundedness of D^δ on $B_p(\beta)$.

Theorem 2. Let $0 < p < \infty$. Then $D^\delta B_p(\beta) \to B_p(\widetilde{\beta})$, where $\widetilde{\beta} = \beta + \delta p$.

Proof. Let $f \in B_p(\beta)$ and $F(z) = D^\delta f(z)$. We show that $F \in B_p(\widetilde{\beta})$. By (6) we have

$$f(z) = C(n,\beta) \int\limits_{B^n} \frac{(1-|\zeta|^2)^\beta}{(1-\langle z,\zeta\rangle)^{n+\beta}} Df(\zeta)d\nu(\zeta),$$

$$|DF(z)| \le C(n,\beta) \int_{B^n} \frac{(1-|\zeta|^2)^\beta g(\langle z,\zeta\rangle)}{|1-\langle z,\zeta\rangle|^{n+1+\beta+\delta}} |Df(\zeta)g(\langle z,\zeta\rangle)|d\nu(\zeta).$$

Using Hölder's inequality, for $p > 1$ we get

$$|DF(z)|^p \le C(n,\beta) \int\limits_{B^n} \frac{(1-|\zeta|^2)^{p\beta}|Df(\zeta)|^p}{|1-\langle z,\zeta\rangle|^{\beta+\delta+n+1}} d\nu(\zeta)\times$$

$$\times \left(\int\limits_{B^n} \frac{d\nu(\zeta)}{|1-\langle z,\zeta\rangle|^{\beta+\delta+n+1}} \right)^{p/q} \le$$

$$\le \frac{C(n,\beta)}{(1-|z|)^{(\beta+\delta)p/q}} \int\limits_{B^n} \frac{(1-|\zeta|^2)^{\beta p}|Df(\zeta)|^p}{|1-\langle z,\zeta\rangle|^{\beta+\delta+n+1}} d\nu(\zeta).$$

In the last inequality we have used Lemma 2. Therefore, changing the integration order and using Lemma 2 again, we get

$$\|F\|_{B_p(\widetilde{\beta})} = \int\limits_{B^n} \frac{|DF(z)|^p d\nu(z)}{(1-|z|^2)^{n+1-p-\widetilde{\beta}}} =$$

$$= C(n,\beta) \int\limits_{B^n} \frac{(1-|\zeta|^2)^{\beta p}|Df(\zeta)|^p d\nu(\zeta)}{(1-|\zeta|^2)^{\beta+\delta-\widetilde{\beta}-p+n+1+(\beta-\delta)p/q}} =$$

$$= C(n,\beta) \int\limits_{B^n} \frac{|Df(\zeta)|^p d\nu(\zeta)}{(1-|\zeta|^2)^{n+1-\beta-p}} = \|f\|_{B_p(\widetilde{\beta})}.$$

Let $0 < p \le 1$. Then, using Lemma 4, we get

$$|DF(z)|^p \preceq C(n,m) \int\limits_{B^n} \frac{(1-|\zeta|^2)^{(n+1)p+mp}|Df(\zeta)|^p}{|1-\langle z,\zeta\rangle|^{(m+n+\delta+1)p}(1-|\zeta|^2)^{n+1}} d\nu(\zeta)$$

for sufficiently large m. Then

$$\int\limits_{B^n} |DF(z)|^p \frac{(1-|z|^{\widetilde{\beta}} d\nu(z)}{(1-|z|^2)^{n+1-p}} = \int\limits_{B^n} |Df(\zeta)|^p (1-|\zeta|^2)^{(n+1)p+mp-n-1} \times$$

$$\times \int\limits_{B^n} \frac{(1-|z|)^{\widetilde{\beta}+p-n-1} d\nu(z)d\nu(\zeta)}{|1-\langle z,\zeta\rangle|^{(m+n+1+\delta)p}} = \int\limits_{B^n} \frac{|Df(\zeta)|^p d\nu(\zeta)}{(1-|\zeta|^2)^{p\delta-p-\widetilde{\beta}+n+1}} = \|f\|_{B_p(\beta)}. \qquad \square$$

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