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Mathematics

ON A REPRESENTATION OF THE RIEMANN ZETA FUNCTION

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In this paper a new representation of the Riemann zeta function in the disc $U(2,1)$ is obtained: $\zeta(z)=\dfrac{1}{z-1}+\sum\limits_{n=0}^{\infty}(-1)^n\alpha_n(z-2)^n$, where the coefficients α_k are real numbers tending to zero. Hence is obtained $\gamma=\lim\limits_{m\to\infty}\left[\sum\limits_{k=0}^{n-1}\dfrac{\zeta^{(k)}(2)}{k!}-n\right]$, where γ is the Euler–Mascheroni constant.

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Introduction. Let us consider the Riemann function

$$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}, \tag{1}$$

where $\operatorname{Re} z>1$.

It is well-known that this function has an analitic continuation in $\mathbb{C}\setminus\{1\}$ and has a simple pole of first order at $z=1$ (see [1]).

There are different representations of the Riemann function. For example:

$$\zeta(z)=\frac{1}{z-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n(z-1)^n,$$

where $\gamma_n=\lim\limits_{m\to\infty}\left[\sum\limits_{k=1}^{m}\dfrac{(\ln k)^n}{k}-\dfrac{(\ln m)^{n+1}}{n+1}\right]$ are the Stieltjes coefficients (see [2]).

In this paper a new representation of the Riemann function in the disc $U(2,1)$ is obtained: $\zeta(z)=\dfrac{1}{z-1}+\sum\limits_{n=0}^{\infty}(-1)^n\alpha_n(z-2)^n$, where the coefficients α_k are real vanishing numbers. We obtain also the following formula:

$$\gamma=\lim_{n\to\infty}\left[\sum_{k=0}^{n-1}\frac{\zeta^{(k)}(2)}{k!}-n\right], \tag{2}$$

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where γ is the Euler–Mascheroni constant.

In the proof we will use the following well-known relation (see, e.g., [2]):

$$\gamma = \lim_{z\to 1}\left[\zeta(z) - \frac{1}{z-1}\right]. \tag{3}$$

The Main Result. Consider the function $\zeta(z)$ in $U(2,1)$, where

$$U(2,1) = \{z \in \mathbb{C} : |z-2| < 1\}.$$

Below we present our main result.

T e o r e m 1 . The Riemann zeta function can be represented in $U(2,1)$ by

$$\zeta(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty}(-1)^n \alpha_n (z-2)^n, \tag{4}$$

where the coefficients α_k are real numbers tending to zero.

P r o o f . Let us represent the Riemann zeta function by power series in the neighborhood of $z = 2$. We obtain that for all $z \in U(2,1)$

$$\zeta(z) = \sum_{n=0}^{\infty}(-1)^n c_n (z-2)^n, \tag{5}$$

where $c_0 = \zeta(2) = \dfrac{\pi^2}{6}, \quad (-1)^n c_n = \dfrac{\zeta^{(n)}(2)}{n!}, \quad c_n = \dfrac{1}{n!}\displaystyle\sum_{k=2}^{\infty}\frac{\ln^n k}{k^2}.$

Denote $\alpha_n = c_n - 1, \quad n = 0,1,2...$

We have

$$\zeta(z) = \sum_{n=0}^{\infty}(-1)^n[1 + (c_n - 1)](z-2)^n = \frac{1}{z-1} + \sum_{n=0}^{\infty}(-1)^n \alpha_n (z-2)^n.$$

Thus we get (4) and it remains to prove that

$$\lim_{n\to\infty} c_n = 1. \tag{6}$$

We have that the function $f(x) = \dfrac{\ln^n x}{x^2}$ is a monotonic and decreasing for $x \geq \exp(n/2)$.

Denote $m := [\exp(n/2)]$. In the case when $k-1 \geq m+1$ we have

$$\int_{k-1}^{k} f(x)dx \geq \frac{\ln^n k}{k^2} \geq \int_{k}^{k+1} f(x)dx. \tag{7}$$

Summing the values in (7) for $k \geq m+2$, we obtain

$$\int_{m+1}^{+\infty} f(x)dx \geq \sum_{k=m+2}^{\infty}\frac{\ln^n k}{k^2} \geq \int_{m+2}^{+\infty} f(x)dx. \tag{8}$$

From here we get

$$\frac{1}{n!}\int_{2}^{+\infty} f(x)dx + B_n \geq c_n \geq \frac{1}{n!}\int_{2}^{+\infty} f(x)dx + D_n, \tag{9}$$

where $B_n = \dfrac{1}{n!}\left(\displaystyle\sum_{k=2}^{m+1}\frac{\ln^n k}{k^2} - \int_{2}^{m+1} f(x)dx\right); D_n = \dfrac{1}{n!}\left(\displaystyle\sum_{k=2}^{m+1}\frac{\ln^n k}{k^2} - \int_{2}^{m+2} f(x)dx\right).$

Let us estimate the quantities B_n and D_n.

$$B_n = \frac{1}{n!}\left(\sum_{k=2}^{m-1}\frac{\ln^n k}{k^2} - \int_2^m f(x)dx + \frac{\ln^n m}{m^2} + \frac{\ln^n(m+1)}{(m+1)^2} - \int_m^{m+1} f(x)dx\right) \leq$$

$$\leq \frac{1}{n!}\left(\sum_{k=2}^{m-1}\left(\frac{\ln^n k}{k^2} - \int_k^{k+1} f(x)dx\right) + \frac{\ln^n m}{m^2} + \frac{\ln^n(m+1)}{(m+1)^2} - \frac{\ln^n m}{(m+1)^2}\right) <$$

$$< \frac{1}{n!}\left(\frac{\ln^n m}{m^2} + \frac{\ln^n(m+1)}{(m+1)^2}\right) =: \beta_n.$$

$$D_n = \frac{1}{n!}\left(\sum_{k=2}^{m-1}\left[\frac{\ln^n k}{k^2} - \int_k^{k+1} f(x)dx\right] + \frac{\ln^n m}{m^2} + \frac{\ln^n(m+1)}{(m+1)^2} - \int_m^{m+2} f(x)dx\right) \geq$$

$$\geq \frac{1}{n!}\left(\sum_{k=2}^{m-1}\left[\frac{\ln^n k}{k^2} - \frac{\ln^n(k+1)}{(k+1)^2}\right] + \frac{\ln^n m}{m^2} + \frac{\ln^n(m+1)}{(m+1)^2} - \frac{\ln^n(m+2)}{m^2}\right) =$$

$$= \frac{1}{n!}\left(\frac{\ln^n 2}{4} + \frac{\ln^n(m+1)}{(m+1)^2} - \frac{\ln^n(m+2)}{m^2}\right) =: \delta_n.$$

Thus, (9) we get from

$$\frac{1}{n!}\int_2^{+\infty} f(x)dx + \beta_n \geq c_n \geq \frac{1}{n!}\int_2^{+\infty} f(x)dx + \delta_n. \tag{10}$$

By using the relation $\lim_{n\to\infty}\frac{\ln^n(\exp(n/2)+b)}{n!\,(\exp(n/2)+a)} = 0$ for any a,b, we obtain that the sequences β_n and δ_n tend to 0.

Hence to complete the proof, in view of (10), it suffices to show that

$$\frac{1}{n!}A_n \to 1,$$

where $A_n = \int_a^\infty f(x)dx = \int_a^\infty \frac{\ln^n x}{x^2}dx, \quad a = 2$. Now let us evaluate

$$A_1 = \int_a^\infty \frac{\ln x}{x^2}dx = \frac{\ln a}{a} + \frac{1}{a},\ A_2 = \int_a^\infty \frac{\ln^2 x}{x^2}dx = \frac{\ln^2 a}{a} + 2A_1, \dots,$$

$$A_n = \int_a^\infty \frac{\ln^n x}{x^2}dx = \frac{\ln^n a}{a} + nA_{n-1}.$$

Therefore we have

$$\frac{A_n}{n!} = \frac{1}{a}\left(1 + \ln a + \frac{\ln^2 a}{2!} + \dots + \frac{\ln^n a}{n!}\right), \tag{11}$$

where $n = 1, 2, \dots$ The right hand side of (11) tends to $\frac{1}{a}\exp(\ln a) = 1$.

Now we get the following

Corollary. We have the following representation for the Euler–Mascheroni constant:

$$\gamma = \lim_{n\to\infty}\left[\sum_{k=0}^{n-1}\frac{\zeta^{(k)}(2)}{k!} - n\right]. \tag{12}$$

P r o o f . We have that the function $\zeta(z)$ is analytic in $\mathbb{C}\setminus\{1\}$ and $z=1$ is a simple pole. Therefore, the second summand in the right hand side of (4) is an entire function. Now, by taking into account (3), we get from (4) that

$$\gamma=\sum_{k=0}^{\infty}\alpha_k,$$

which implies (12). □

Szegó has proved the following [3, 4].

T h e o r e m 2 . Suppose we have the representation $f(z)=\sum\limits_{n=0}^{\infty} f_n z^n$ in a neighborhood of the origin, where the coefficients f_n are bounded and have finitely many limit points $d_1,\dots,d_k$. Then, there exist functions $g(z)$ and $h(z)$ such that

$$f(z)=g(z)+h(z),\quad g(z)=\sum_{n=0}^{\infty} g_n z^n,\quad h(z)=\sum_{n=0}^{\infty} h_n z^n, \tag{13}$$

where each coefficient g_n takes one of values $d_1,\dots,d_k$ and $h_n\to 0$ when $n\to\infty$.

Notice that, in view of (6), the Riemann zeta function $f(z)=\zeta(z)$ satisfies the conditions of Theorem 2 in the neighborhood of $z=2:\ U(2,1)$. Here $f_n=(-1)^n c_n$ and therefore $k=2$ and $d_1=-1, d_2=1$. Therefore, the representation (4) is a special case of the Szegó representation (13), where the functions $g(z)$ and $h(z)$ are presented explicitly:

$$g(z)=\frac{1}{z-1}=\sum_{n=0}^{\infty}(-1)^n(z-2)^n,\quad h(z)=\sum_{n=0}^{\infty}(-1)^n\alpha_n(z-2)^n.$$

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