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Mathematics

ON INTEGRAL OPERATORS OF BERGMAN TYPE ON THE UNIT BALL OF $\mathbb{R}^n$

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We prove the boundedness of Bergman type integral operators in mixed norm spaces over the unit ball of $\mathbb{R}^n$. Bounded harmonic projections are found in the mixed norm and Lipschitz spaces. Corresponding Forelli–Rudin type theorems are proved.

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Introduction. Let $B = B_n$ be an open unit ball in $\mathbb{R}^n$ $(n \geq 2)$ and $S = \partial B$ be its boundary in unit sphere. The integral means of order p of a function $f(x) = f(r\zeta)$ on the sphere $|x| = r$ are denoted by

$$M_p(f;r) := \left\|f(r\cdot)\right\|_{L^p(S;d\sigma)}, \quad 0 \leq r < 1, \quad 0 < p \leq \infty,$$

where $d\sigma$ is the $(n-1)$-dimensional area-surface Lebesgue measure on S normalized, so that $\sigma(S) = 1$. The set of all (real) harmonic functions in the unit ball B is denoted by $h(B)$. Let dV be the Lebesgue volume measure on B normalized, so that $V(B) = 1$. In the polar coordinates we have $dV(x) = n\,r^{n-1}dr\, d\sigma(\zeta)$.

By the definition, the mixed norm space $L(p,q,\alpha)\big(0 < p,q \leq \infty, \alpha \in \mathbb{R}\big)$ is the set of those functions $f(x)$ measurable in the unit ball B, for which the quasi-norm

$$\|f\|_{L(p,q,\alpha)} := \begin{cases} \left(\displaystyle\int_0^1 (1-r)^{\alpha q-1} M_p^q(u;r)dr\right)^{1/q}, & 0 < q < \infty, \\ \operatorname*{ess\,sup}_{0<r<1} (1-r)^{\alpha} M_p(f;r), & q = \infty, \end{cases}$$

is finite. For the subspace of $L(p,q,\alpha)$ consisting of harmonic functions let

$$h(p,q,\alpha) := h(B) \cap L(p,q,\alpha).$$

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The mixed norm spaces $h(p,q,\alpha)$ and their analogues consisting of holomorphic, pluriharmonic or harmonic functions in the disc, the ball in $\mathbb{C}^n$ or $\mathbb{R}^n$ are extensively discussed in the last three decades. The mixed norm spaces of holomorphic functions in the unit disc were introduced by Hardy and Littlewood [1,2] and developed later by Flett [3]. For $p = q < \infty$ the spaces $h(p,q,\alpha)$ coincide with weighted Bergman spaces, see [4,5], while for $q = \infty$ these spaces are referred to as weighted Hardy spaces. The spaces $h(p,p,\alpha), h(p,q,\alpha)$ on the unit ball in $\mathbb{R}^n$ were studied in [6–16], while the space $h(p,q,\alpha)$ consisting of n-harmonic functions on a polydisc in $\mathbb{C}^n$ were studied in [17,18].

In the recent paper we established a reproducing integral formula of Poisson–Bergman type for functions in $h(p,q,\alpha)$ [16].

Theorem A. Let $\alpha > 0$ and $u(x) \in h(p,q,\alpha)$ be an arbitrary function. If either $0 < p,\ q \leq \infty$, $\beta > \max\{\alpha + (n-1)(1/p - 1),\ \alpha\}$ or $1 \leq p \leq \infty,\ 0 < q \leq 1$, $\beta \geq \alpha$, then

$$u(x) = \frac{2}{n\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1} P_\beta(x,y)\, u(y)\, dV(y), \qquad x \in B, \tag{1}$$

where P_β is the Poisson–Bergman type kernel defined below (see Sec. 2).

Integral Eq. (1) induces a linear integral operator of Bergman type

$$T_\beta(u)(x) := \frac{2}{n\Gamma(\beta)} \int_B (1-|y|^2)^{\beta-1} P_\beta(x,y)\, u(y)\, dV(y), \qquad x \in B. \tag{2}$$

In fact, Theorem A asserts that operator T_β is the identity map on $h(p,q,\alpha)$ for suitable parameters, that is, $T_\beta(u) = u,\ \forall\, u \in h(p,q,\alpha)$.

Along with the operator T_β, define more general operators of Bergman type:

$$(T_{\beta,\lambda} f)(x) := \frac{2(1-|x|^2)^\lambda}{n\Gamma(\beta+\lambda)} \int_B (1-|y|^2)^{\beta-1} P_{\beta+\lambda}(x,y)\, f(y)\, dV(y),$$

$$(S_{\beta,\lambda} f)(x) := \frac{2(1-|x|^2)^\lambda}{n\Gamma(\beta+\lambda)} \int_B (1-|y|^2)^{\beta-1} |P_{\beta+\lambda}(x,y)|\, f(y)\, dV(y).$$

Note that $T_{\beta,0} \equiv T_\beta$.

It is natural here to ask whether these operators are bounded in mixed norm spaces.

Theorem 1. If $1 \leq p,q \leq \infty$, $\beta > \alpha > -\lambda$, then the operators $T_{\beta,\lambda}, S_{\beta,\lambda}$ continuously map the space $L(p,q,\alpha)$ into itself:

$$T_{\beta,\lambda},\ S_{\beta,\lambda} : L(p,q,\alpha) \xrightarrow{\text{into}} L(p,q,\alpha), \tag{3}$$

that is,

$$\|T_{\beta,\lambda} f\|_{L(p,q,\alpha)} \leq C \|f\|_{L(p,q,\alpha)}, \qquad f \in L(p,q,\alpha), \tag{4}$$

$$\|S_{\beta,\lambda} f\|_{L(p,q,\alpha)} \leq C \|f\|_{L(p,q,\alpha)}, \qquad f \in L(p,q,\alpha), \tag{5}$$

where $C = C(p,q,\alpha,\beta,\lambda)$ is a positive constant depending only on the indicated parameters. Moreover, the operator $T_{\beta,0}$ continuously projects $L(p,q,\alpha)$ onto $h(p,q,\alpha)$:

$$T_{\beta,0} : L(p,q,\alpha) \xrightarrow{\text{onto}} h(p,q,\alpha). \tag{6}$$

Remark 1. Theorem 1 is an analogue of the well-known Forelli–Rudin theorem in Bergman spaces. Various generalizations for holomorphic and harmonic functions can also be found in [4–10, 17].

Preliminaries and Proof of Theorem 1. Throughout the paper, we always assume $x = r\zeta$, $y = \rho\eta$, $0 \le r, \rho < 1$, $\zeta, \eta \in S$, and the letters $C(\alpha, \beta, \dots), C_\alpha$ etc. stand for different positive constants depending only on the indicated parameters.

Definition 1. (Riemann–Liouville fractional integral and derivative for $\mathbb{R}^2$). For a function $f(r)$ of one variable $r \in [0,1)$, let

$$D^{-\alpha} f(r) := \frac{1}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} f(t)\, dt = \frac{r^\alpha}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tr)\, dt,$$

$$D^m f(r) := \left(\frac{d}{dr}\right)^m f(r), \qquad D^\alpha f(r) := D^{-(m-\alpha)} D^m f(r),$$

where $0 \le r < 1$, $\alpha > 0$, $m \in \mathbb{Z}$, $m \ge 0$, $m - 1 < \alpha \le m$.

Definition 2. (Fractional integral and derivative for $\mathbb{R}^n$, $n \ge 2$).

Given a function $f(x)$ in the unit ball B, let

$$\mathcal{D}_n^{-\alpha} f(x) := r^{-(\alpha+n/2-1)} D^{-\alpha}\left\{r^{n/2-1} f(x)\right\} =$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tx)\, t^{n/2-1}\, dt,$$

$$\mathcal{D}_n^{\alpha} f(x) := r^{-(n/2-1)} D^{\alpha}\left\{r^{\alpha+n/2-1} f(x)\right\}, \qquad r = |x|.$$

This version of the fractional derivative in $\mathbb{R}^n$ is introduced in [16] and makes it possible to apply it to the extended Poisson kernel in B [13]

$$P(x,y) \equiv P_0(x,y) := \frac{1 - |x|^2|y|^2}{(1 - 2x\cdot y + |x|^2|y|^2)^{n/2}}, \qquad x \in B,\ y \in \overline{B},$$

in order to obtain the Poisson–Bergman type kernel P_β in B mentioned in Theorem A. Here $x \cdot y$ means the Euclidean inner product.

Definition 3. (Poisson–Bergman type kernel in B).

$$P_\alpha(x,y) := \mathcal{D}_n^\alpha P(x,y) \quad x, y \in B, \quad \alpha \ge 0. \tag{7}$$

Similar kernels are defined in [11] (for integer α), [4, 6, 7, 9, 10, 12].

Note that Poisson–Bergman type kernel $P_\beta(x,y)$ is a harmonic function with respect to both x and y.

Lemma 1. For any $\beta > \alpha > 0$ there hold the inequalities

$$\int_S \frac{d\sigma(\xi)}{|\xi - x|^{\alpha+n-1}} \le C(\alpha, n) \frac{1}{(1-|x|)^\alpha}, \qquad x \in B, \tag{8}$$

$$\int_0^1 \frac{(1-t)^{\alpha-1}}{(1-rt)^\beta} dt \le C(\alpha, \beta) \frac{1}{(1-r)^{\beta-\alpha}}, \qquad 0 \le r < 1, \tag{9}$$

$$\int_B \frac{(1-|y|)^{\alpha-1}}{|\zeta - ry|^{\beta+n-1}}\, dV(y) \le C(\alpha, \beta, n) \frac{1}{(1-|x|)^{\beta-\alpha}}, \qquad x = r\zeta \in B. \tag{10}$$

The estimates of Lemma 1 are well known and can be found in [7, 12, 14].

We also need the next well-known inequality (see, e.g., [3]).

Lemma 2. (Hardy's inequality). If $1 \le p < \infty$, $\beta > -1$, $g(r) \ge 0$, then

$$\int_0^1 (1-r)^\beta \left(\int_0^r g(t)dt \right)^p dr \le C \int_0^1 (1-r)^{\beta+p} g^p(r) dr,$$

where the constant $C = C(p,\beta) > 0$ depends only on p and β.

Proof of Theorem 1. Let $f(x) \in L(p,q,\alpha)$ be an arbitrary function. It suffices to prove only (5).

We need an estimate for the Poisson–Bergman type kernel P_β (see, e.g., [7, 8, 12, 14])

$$|P_\alpha(x,y)| \le \frac{C(\alpha,n)}{|\rho x - \eta|^{\alpha+n-1}}, \qquad x = r\zeta, \quad y = \rho\eta, \qquad \alpha \ge 0,$$

It follows that

$$\left|(S_{\beta,\lambda} f)(x)\right| \le C(\beta,\lambda,n)\,(1-|x|^2)^\lambda \int_B (1-|y|^2)^{\beta-1}\,|P_{\beta+\lambda}(x,y)|\,|f(y)|\,dV(y) \le$$

$$\le C(\beta,\lambda,n)\,(1-|x|^2)^\lambda \int_B \frac{(1-\rho^2)^{\beta-1}}{|\rho x - \eta|^{\beta+\lambda+n-1}}\,|f(y)|\,\rho^{n-1} d\rho\, d\sigma(\eta).$$

Replace here x by Qx, where Q is an arbitrary orthogonal linear transformation $Q : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, that is, $|Qx| = |x|$ for all $x \in \mathbb{R}^n$. Recall that the measure σ is invariant under rotations, meaning $\sigma\big(Q(G)\big) = \sigma(G)$ for every Borel set $G \subset S$ and every orthogonal transformation Q. Applying also the change $\eta \mapsto Q\eta$, we find that

$$\left|(S_{\beta,\lambda} f)(Qx)\right| \le$$

$$\le C(\beta,\lambda,n)\,(1-|Qx|^2)^\lambda \int_B \frac{(1-\rho^2)^{\beta-1}}{|\rho Qx - Q\eta|^{\beta+\lambda+n-1}}\,|f(\rho Q\eta)|\,\rho^{n-1} d\rho d\sigma(\eta) =$$

$$= C(\beta,\lambda,n)\,(1-|x|^2)^\lambda \int_0^1 \int_S \frac{(1-\rho^2)^{\beta-1}}{|\rho x - \eta|^{\beta+\lambda+n-1}}\,|f(\rho Q\eta)|\,\rho^{n-1} d\rho d\sigma(\eta).$$

Further, we use Minkowski's inequality in the continuous form, Lemma 1 and the identity

$$M_p(F;|z|) = \left(\int |F(Qz)|^p dQ \right)^{1/p}, \qquad z \in B,$$

where the integral is taken over the orthogonal group. Hence,

$$M_p(S_{\beta,\lambda} f; r) \le$$

$$\le C(\beta,\lambda,n)\,(1-|x|^2)^\lambda \int_0^1 \int_S \frac{(1-\rho^2)^{\beta-1}}{|\rho x - \eta|^{\beta+\lambda+n-1}}\, M_p(f;\rho)\, d\rho d\sigma(\eta) \le$$

$$\le C(\beta,\lambda,n)\,(1-r^2)^\lambda \int_0^1 \frac{(1-\rho^2)^{\beta-1}}{(1-r\rho)^{\beta+\lambda}}\, M_p(f;\rho)\, d\rho = \tag{11}$$

$$= C(\beta,\lambda,n)\,(1-r^2)^\lambda \left(\int_0^r + \int_r^1 \right) \frac{(1-\rho^2)^{\beta-1}}{(1-r\rho)^{\beta+\lambda}}\, M_p(f;\rho)\, d\rho \le$$

$$\le C\,(1-r^2)^\lambda \int_0^r \frac{M_p(f;\rho)}{(1-\rho)^{1+\lambda}}\, d\rho + C\,(1-r)^{-\beta} \int_r^1 (1-\rho^2)^{\beta-1} M_p(f;\rho)\, d\rho.$$

C a s e $1 \le q < \infty$. First note that the last integral in (11) is convergent since $\beta > \alpha$ and

$$\int_0^1 (1-\rho^2)^{\beta-1} M_p(f;\rho)\, d\rho \le \int_0^1 (1-\rho^2)^{\alpha-1} M_p(f;\rho)\, d\rho \le C_\alpha \|f\|_{L(p,1,\alpha)} < +\infty$$

for $q = 1$, while for $1 < q < \infty$ we estimate by Hölder's inequality $(1/q + 1/q' = 1)$

$$\int_0^1 (1-\rho^2)^{\beta-1} M_p(f;\rho)\, d\rho = \int_0^1 (1-\rho^2)^{\beta-\alpha+\alpha} M_p(f;\rho) \frac{d\rho}{1-\rho} \le$$
$$\le C_\beta \left(\int_0^1 (1-\rho)^{q'(\beta-\alpha)} \frac{d\rho}{1-\rho} \right)^{1/q'} \left(\int_0^1 (1-\rho)^{\alpha q} M_p^q(f;\rho) \frac{d\rho}{1-\rho} \right)^{1/q} =$$
$$= C(\alpha,\beta,q) \|f\|_{L(p,q,\alpha)} < +\infty.$$

By the triangle inequality and next by Hardy's inequality in Lemma 2,

$$\|S_{\beta,\lambda} f\|_{L(p,q,\alpha)} = \left\|(1-r)^\alpha M_p(S_{\beta,\lambda} f;r)\right\|_{L^q(dr/(1-r))} \le$$
$$\le C \left\| (1-r)^{\alpha+\lambda} \int_0^r M_p(f;\rho) \frac{d\rho}{(1-\rho)^{1+\lambda}} \right\|_{L^q(dr/(1-r))} +$$
$$+ C \left\| (1-r)^{\alpha-\beta} \int_r^1 (1-\rho)^{\beta-1} M_p(f;\rho) d\rho \right\|_{L^q(dr/(1-r))} \le$$
$$\le C \left[\int_0^1 (1-r)^{(\alpha+\lambda)q-1} \left(\frac{1-r}{(1-r)^{1+\lambda}} M_p(f;r) \right)^q dr \right]^{1/q} +$$
$$+ C \left[\int_0^1 \rho^{(\alpha-\beta)q-1} \left(\int_0^\rho t^{\beta-1} M_p(f;1-t) dt \right)^q d\rho \right]^{1/q} \le C \|f\|_{L(p,q,\alpha)}.$$

C a s e $q = \infty$. Since $(1-r)^\alpha M_p(f;r) \le \|f\|_{L(p,\infty,\alpha)}$ and $\beta > \alpha$, the last integral in (11) again converges

$$\int_0^1 (1-\rho^2)^{\beta-1} M_p(f;\rho)\, d\rho \le C_\beta \int_0^1 (1-\rho)^{\beta-1} \frac{\|f\|_{L(p,\infty,\alpha)}}{(1-\rho)^\alpha}\, d\rho =$$
$$= C(\alpha,\beta) \|f\|_{L(p,\infty,\alpha)} < +\infty.$$

Therefore by Lemma 1,

$$M_p(S_{\beta,\lambda} f;r) \le C(\beta,\lambda,n)\, (1-r^2)^\lambda \int_0^1 \frac{(1-\rho)^{\beta-1}}{(1-r\rho)^{\beta+\lambda}} M_p(f;\rho)\, d\rho \le$$
$$\le C(\beta,\lambda,n)\, (1-r^2)^\lambda \int_0^1 \frac{(1-\rho^2)^{\beta-1}}{(1-r\rho)^{\beta+\lambda}} \frac{\|f\|_{L(p,\infty,\alpha)}}{(1-\rho)^\alpha}\, d\rho =$$
$$= C(\beta,\lambda,n)\, \|f\|_{L(p,\infty,\alpha)}\, (1-r^2)^\lambda \int_0^1 \frac{(1-\rho)^{\beta-\alpha-1}}{(1-r\rho)^{\beta+\lambda}}\, d\rho \le$$
$$\le C(\alpha,\beta,\lambda,n)\, \|f\|_{L(p,\infty,\alpha)}\, (1-r^2)^\lambda \frac{1}{(1-r)^{\alpha+\lambda}}.$$

Thus

$$(1-r)^\alpha M_p(S_{\beta,\lambda} f;r) \le C(\alpha,\beta,\lambda,n) \left\|f\right\|_{L(p,\infty,\alpha)}, \quad 0 \le r < 1,$$

so, (4) and (5) are proved.

Since the operator $T_{\beta,0}$ ($\lambda = 0$) is bounded on $L(p,q,\alpha)$ and is the identity map on $h(p,q,\alpha)$ (by Theorem A), the mapping (6), $T_{\beta,0} : L(p,q,\alpha) \longrightarrow h(p,q,\alpha)$ is a harmonic projection of $L(p,q,\alpha)$ onto $h(p,q,\alpha)$. The proof of Theorem 1 is complete. □

Bounded Projection on the Lipschitz Space. It would be of interest to find out the images of other classical function spaces under the Bergman operators. It turns out that T_β preserves the Lipschitz classes in B.

Definition 4. A function $f(x)$ given in the unit ball B is said to belong to Lipschitz space $\mathrm{Lip}\,\alpha$ $(0<\alpha<1)$, if

$$|f(x)-f(y)| \le C(\alpha,n)|x-y|^\alpha, \qquad x,y\in B.$$

The Lipschitz space $\mathrm{Lip}\,\alpha$ is equipped with a seminorm

$$\|f\|_{\mathrm{Lip}\,\alpha} := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha},$$

and $\mathrm{Lip}\,\alpha$ becomes a Banach space with the norm $|f(0)| + \|f\|_{\mathrm{Lip}\,\alpha}$. Let $\mathrm{hLip}\,\alpha$ be the subspace of $\mathrm{Lip}\,\alpha$ consisting of harmonic functions,

$$\mathrm{hLip}\,\alpha := h(B)\cap \mathrm{Lip}\,\alpha.$$

The following is an analogue of the classical Hardy–Littlewood theorem [1]. The proof is a repetition of the classical one with obvious changes.

Lemma 3. Harmonic function $u(x)$ is in $\mathrm{hLip}\,\alpha$ $(0<\alpha<1)$ if and only if

$$|\nabla u(x)| \le C(\alpha,n)(1-|x|)^{\alpha-1}, \qquad x\in B.$$

Corresponding seminorms are equivalent:

$$C(\alpha,n)\|u\|_{\mathrm{Lip}\,\alpha} \le \sup_{x\in B}(1-|x|)^{1-\alpha}|\nabla u(x)| \le C(\alpha,n)\|u\|_{\mathrm{Lip}\,\alpha}.$$

Now we study the action of the Bergman type operators in Lipschitz spaces.

Theorem 2. For $0<\alpha<1$, $\beta>0$, the operator T_β continuously projects $\mathrm{Lip}\,\alpha$ onto its harmonic subspace $\mathrm{hLip}\,\alpha$,

$$T_\beta : \mathrm{Lip}\,\alpha \xrightarrow{\text{onto}} \mathrm{hLip}\,\alpha,$$

that is,

$$\|T_\beta f\|_{\mathrm{hLip}\,\alpha} \le C(\alpha,\beta,n)\|f\|_{\mathrm{Lip}\,\alpha}, \qquad f\in \mathrm{Lip}\,\alpha.$$

Proof. Let $f(x)\in \mathrm{Lip}\,\alpha$ be an arbitrary function not necessarily harmonic. Because of Theorem A

$$1 = \frac{2}{n\Gamma(\beta)}\int_B (1-|y|^2)^{\beta-1} P_\beta(x,y)\,dV(y), \qquad x\in B.$$

It follows that for any fixed point $z\in B$, $0<|z-x|<1-|x|$,

$$T_\beta(f)(x) = \frac{2}{n\Gamma(\beta)}\int_B (1-|y|^2)^{\beta-1} P_\beta(x,y)\left(f(y)-f(z)\right)dV(y) + f(z). \tag{12}$$

We also need an estimate for the derivatives of the Poisson–Bergman kernel (see, e.g., [7, 8, 12, 14])

$$|\nabla_x P_\alpha(x,y)| \le \frac{C(\alpha,n)}{|\rho x-\eta|^{\alpha+n}}, \qquad x=r\zeta, \quad y=\rho\eta, \qquad \alpha\ge 0, \tag{13}$$

Differentiation of (12) with the gradient ∇_x and estimation by using of (13) and the obvious inequality $|y-x|<|\rho x-\eta|$, $x,y\in B$, lead to

$$\left|\nabla_x T_\beta(f)(x)\right| \le \frac{2}{n\Gamma(\beta)}\int_B (1-|y|^2)^{\beta-1}\left|\nabla_x P_\beta(x,y)\right| |f(y)-f(z)|\,dV(y) \le$$
$$\le C(\alpha,\beta,n)\int_B (1-|y|^2)^{\beta-1}\,\|f\|_{\mathrm{Lip}\,\alpha}\frac{|y-z|^\alpha}{|\rho x-\eta|^{\beta+n}}\,dV(y) \le$$
$$\le C(\alpha,\beta,n)\,\|f\|_{\mathrm{Lip}\,\alpha}\int_B \frac{|y-z|^\alpha}{|\rho x-\eta|^{n+1}}\,dV(y).$$

The application of the triangle inequality and Lemma 1 finally yields:

$$\left|\nabla_x T_\beta(f)(x)\right| \le C\|f\|_{\mathrm{Lip}\,\alpha}\int_B \frac{|y-x|^\alpha}{|\rho x-\eta|^{n+1}}\,dV(y)+$$
$$+C\|f\|_{\mathrm{Lip}\,\alpha}|x-z|^\alpha\int_B \frac{1}{|\rho x-\eta|^{n+1}}\,dV(y) \le$$
$$\le C\|f\|_{\mathrm{Lip}\,\alpha}\int_B \frac{1}{|\rho x-\eta|^{n+1-\alpha}}\,dV(y)+$$
$$+C\|f\|_{\mathrm{Lip}\,\alpha}(1-|x|)^\alpha\int_B \frac{1}{|\rho x-\eta|^{n+1}}\,dV(y) \le$$
$$\le C\|f\|_{\mathrm{Lip}\,\alpha}\frac{1}{(1-|x|)^{1-\alpha}}+C\|f\|_{\mathrm{Lip}\,\alpha}\frac{(1-|x|)^\alpha}{1-|x|}=$$
$$=C(\alpha,\beta,n)\|f\|_{\mathrm{Lip}\,\alpha}\frac{1}{(1-|x|)^{1-\alpha}}.$$

This together with Lemma 3 completes the proof of Theorem 2. □

Remark 2. Theorem 2 asserts that the Bergman operator $T_\beta\equiv T_{\beta,0}$ acts as a bounded harmonic projection in Lipschitz spaces. Preservation of Lipschitz spaces under the Bergman projection was studied in [19] (for unweighted Bergman projection) and in [20] (for integer β).

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