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I n f o r m a t i c s

ON TYPED AND UNTYPED LAMBDA-TERMS

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Typed λ-terms that use variables of any order and don't use constants of order > 1 are studied in the paper. Used constants of order 1 are strong computable functions and each of them has an untyped λ-term, which λ-defines it. Besides, for the set of built-in functions there exists such a notion of δ-reduction, that every typed term has a single normal form. An algorithm of translation of typed λ-terms to untyped λ-terms is presented. According to that algorithm, each typed term t is mapped to an untyped term t'. We study in which case typed terms t_1, t_2 such that $t_1 \to\to_{\beta\delta} t_2$ correspond to untyped terms $t_1{}', t_2{}'$ such that $t_1{}' \to\to_{\beta} t_2{}'$.

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1. Typed λ-terms. The following definitions are taken from [1]. Let M be a partially ordered set, which has a least element $\perp$ and every element of M is comparable with itself and with $\perp$. Let $Types$ be the following set:

- $M \in Types$;
- if $\beta, \alpha_1, \dots, \alpha_k \in Types$ $(k > 0)$, then the set of all monotonic mappings from $\alpha_1 \times \dots \times \alpha_k$ to β (denoted by $[\alpha_1 \times \dots \times \alpha_k \to \beta]$) belongs to $Types$.

Let $\alpha \in Types$ and V^T_α be a countable set of variables of type α, then $V^T = \cup_{\alpha \in Types} V^T_\alpha$ is the set of all variables. The set of all terms, denoted by $\Lambda^T = \cup_{\alpha \in Types} \Lambda^T_\alpha$, where Λ^T_α is the set of terms of type α, is defined the following way:

- if $c \in \alpha$, $\alpha \in Types$, then $c \in \Lambda^T_\alpha$;
- if $x \in V^T_\alpha, \alpha \in Types$, then $x \in \Lambda^T_\alpha$;
- if $\tau \in \Lambda^T_{[\alpha_1 \times \dots \times \alpha_n \to \beta]}$, $t_i \in \Lambda^T_{\alpha_i}, i = 1, \dots, n$ $(n \geq 1)$, where $\alpha_1, \dots, \alpha_n$, $\beta \in Types$, then $\tau(t_1, \dots, t_n) \in \Lambda^T_\beta$. The term $\tau(t_1, \dots, t_n)$ is said to be obtained by the operation of application;

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• if $\tau \in \Lambda^T_\beta, x_i \in V^T_{\alpha_i}$, where $\alpha_1, \ldots, \alpha_n, \beta \in Types$ and $i \neq j \Rightarrow x_i \neq x_j$, $i, j = 1, \ldots, n$ $(n \geq 1)$, then $\lambda x_1 \ldots x_n[\tau] \in \Lambda^T_{[\alpha_1 \times \ldots \times \alpha_n \to \beta]}$. The term $\lambda x_1 \ldots x_n[\tau]$ is said to be obtained by the operation of abstraction.

The notions of free and bound occurrences of variables in typed terms as well as the notion of free variable are introduced in the conventional way. The set of all free variables of a typed term t is denoted by $FV(t)$. A term which doesn't contain free variables is called a closed term. Typed terms t_1 and t_2 are said to be congruent (which is denoted by $t_1 \equiv t_2$), if one term can be obtained from the other by renaming bound variables. In what follows, congruent terms are considered identical.

Let $t \in \Lambda^T_\alpha$, $\alpha \in Types$ and $FV(t) \subset \{y_1, \ldots, y_n\}$, $\bar{y_0} = \langle y^0_1, \ldots, y^0_n \rangle$, where $y_i \in V^T_{\beta_i}$, $y^0_i \in \beta_i$, $\beta_i \in Types$, $i = 1, \ldots, n$, $n \geq 0$. The value of the term t for the values of the variables $y_1, \ldots, y_n$ equal to $\bar{y_0}$, is denoted by $Val_{\bar{y_0}}(t)$ and defined as follows:

• if $t \equiv c$ and $c \in \alpha$, then $Val_{\bar{y_0}}(c) = c$;

• if $t \equiv x, x \in V^T_\alpha$, then $Val_{\bar{y_0}}(x) = y^0_i$, where $FV(x) = \{x\} \subset \{y_1, \ldots, y_n\}$ and $x \equiv y_i$, $i = 1, \ldots, n$, $n \geq 0$;

• if $t \equiv \tau(t_1, ..., t_k) \in \Lambda^T_\alpha$, where $\tau \in \Lambda^T_{[\alpha_1 \times \ldots \times \alpha_k \to \alpha]}, t_i \in \Lambda^T_{\alpha_i}$, $\alpha, \alpha_i \in Types$, $i = 1, \ldots, k, k \geq 1$, then $Val_{\bar{y_0}}(\tau(t_1, \ldots, t_k)) = Val_{\bar{y_0}}(\tau)(Val_{\bar{y_0}}(t_1), \ldots, Val_{\bar{y_0}}(t_k))$;

• if $t \equiv \lambda x_1 \ldots x_k[\tau] \in \Lambda^T_\alpha$, where $\alpha = [\alpha_1 \times \ldots \times \alpha_k \to \beta], \tau \in \Lambda^T_\beta, x_i \in V^T_{\alpha_i}$, $\beta, \alpha_i \in Types$, $i = 1, \ldots, k$, $k \geq 1$, then $Val_{\bar{y_0}}(\lambda x_1 \ldots x_k[\tau]) \in [\alpha_1 \times \ldots \times \alpha_k \to \beta]$ and is defined as follows: let $\{y_1, \ldots, y_n\} \setminus \{x_1, \ldots, x_k\} = \{y_{i_1}, \ldots, y_{i_s}\}$, $s \geq 0$, and $\bar{y_1} = \langle y^0_{i_1}, \ldots, y^0_{i_s} \rangle$, then for any $\bar{x_0} =< x^0_1, \ldots, x^0_k >$, $x^0_j \in \alpha_j$, $j = 1, \ldots, k$, $Val_{\bar{y_0}}(\lambda x_1 \ldots x_k[\tau])(x^0_1, \ldots, x^0_k) = Val_{(\bar{x_0}, \bar{y_1})}(\tau)$, where $\bar{x_0}, \bar{y_1} =< x^0_1, \ldots, x^0_k, y^0_{i_1}, \ldots, y^0_{i_s} >$.

It follows from [1] that for any $\bar{y_0} =< y^0_1, \ldots, y^0_n >$, $\bar{y_0} =< y^1_1, \ldots, y^1_n >$ such that $\bar{y_0} \subseteq \bar{y_1}$ and $y^0_i, y^1_i \in \beta_i$ $(1 \leq i \leq n)$ we have:

1. $Val_{\bar{y_0}}(t) \in \alpha$;
2. $Val_{\bar{y_0}}(t) \subseteq Val_{\bar{y_1}}(t)$.

A term obtained by the simultaneous substitution of the terms $t_1, \ldots, t_n$ in the term t for all free occurrences of variables $x_1, \ldots, x_n$ respectively is denoted by $t[x_1 := t_1, \ldots, x_n := t_n]$. A substitution is said to be admissible, if all free variables of the term being substituted remain free after the substitution. We will consider only admissible substitutions.

Let $FV(t_1) \cup FV(t_2) = \{y_1, \ldots, y_n\}$, $y_i \in V^T_{\beta_i}$, $\beta_i \in Types$, $i = 1, \ldots, n$, $n \geq 0$, terms t_1 and t_2 are called equivalent (denoted by $t_1 \sim t_2$), if for any $\bar{y_0} =< y^0_1, \ldots, y^0_n >$, where $y^0_i \in \beta_i$, $i = 1, \ldots, n$, we have the following: $Val_{\bar{y_0}}(t_1) = Val_{\bar{y_0}}(t_2)$. A term $t \in \Lambda^T_\alpha$ is called a constant term with $a \in \alpha$ value, if $t \sim a$.

A term $t \in \Lambda^T$ with a fixed occurrence of a subterm $\tau_1 \in \Lambda^T_\alpha$, where $\alpha \in Types$, is denoted by t_{τ_1}, and a term with this occurrence of τ_1 replaced by τ_2, where $\tau_2 \in \Lambda^T_\alpha$, is denoted by t_{τ_2}.

Let τ_1, τ_2 be terms, t_{τ_1} be a term with a fixed occurrence of the subterm τ_1, then $\tau_1 \sim \tau_2 \Rightarrow t_{\tau_1} \sim t_{\tau_2}$ [2].

A term of the form $\lambda x_1 \ldots x_k[\tau](t_1, \ldots, t_k)$, where $x_i \in V^T_{\alpha_i}, i \neq j \Rightarrow x_i \neq x_j$,

$\tau \in \Lambda^T$, $t_i \in \Lambda^T_{\alpha_i}, \alpha_i \in Types$, $i, j = 1, \dots, k$, $k \geq 1$, is called a β-redex, its convolution is the term $\tau[x_1 := t_1, \dots, x_k := t_k]$. A term t_1 is said to be obtained from a term t_0 by one-step β-reduction (denoted by $t_0 \to_\beta t_1$), if $t_0 \equiv t_{\tau_0}$, $t_1 \equiv t_{\tau_1}, \tau_0$ is a β-redex and τ_1 is its convolution. A term t is said to be obtained from a term t_0 by β-reduction (denoted by $t_0 \to\to_\beta t$), if there exists a finite sequence of terms $t_1, \dots, t_n$ $(n \geq 1)$ such that $t_1 \equiv t_0$, $t_n \equiv t$ and $t_i \to_\beta t_{i+1}$, where $i = 1, \dots, n-1$. A term that contains no β-redexes is called a β-normal form. The set of all β-normal forms is denoted by $\beta - NF^T$.

The definition of δ-redex is taken from [2], a δ-redex has a form $f(t_1, \dots, t_k)$, where $f \in [M^k \to M]$, $t_i \in \Lambda^T_M$, $i = 1, \dots, k$, $k \geq 1$, its convolution is either $m \in M$ and in this case $f(t_1, \dots, t_k) \sim m$, or a subterm t_i, in this case $f(t_1, \dots, t_k) \sim t_i$, $i = 1, \dots, k$. A term t_1 is said to be obtained from a term t_0 by one-step δ-reduction (denoted by $t_0 \to_\delta t_1$), if $t_0 \equiv t_{\tau_0}$, $t_1 \equiv t_{\tau_1}, \tau_0$ is a δ-redex and τ_1 is its convolution. A term t is said to be obtained from a term t_0 by δ-reduction (denoted by $t_0 \to\to_\delta t$), if there exists a finite sequence of terms $t_1, \dots, t_n$ $(n \geq 1)$ such that $t_1 \equiv t_0, t_n \equiv t$ and $t_i \to_\delta t_{i+1}$, where $i = 1, \dots, n-1$.

A term t_1 is said to be obtained from a term t_0 by one-step $\beta\delta$-reduction ($t_0 \to_{\beta\delta} t_1$), if either $t_0 \to_\beta t_1$ or $t_0 \to_\delta t_1$. A term t is said to be obtained from a term t_0 by $\beta\delta$-reduction ($t_0 \to\to_{\beta\delta} t$), if there exists a finite sequence of terms $t_1, \dots, t_n$ $(n \geq 1)$ such that $t_1 \equiv t_0$, $t_n \equiv t$ and $t_i \to_{\beta\delta} t_{i+1}$, where $i = 1, \dots, n-1$. A term containing no $\beta\delta$-redexes is called a normal form. The set of all normal forms is denoted by NF^T.

Let t_1, t_2 be terms, then $t_1 \to\to_{\beta\delta} t_2 \Rightarrow t_1 \sim t_2$ [2]. A fixed set of term pairs (τ_0, τ_1), where τ_0 is δ-redex and τ_1 is its convolution, is called a notion of δ-reduction and is denoted by δ. A notion of δ-reduction is called natural, if:

1. δ is a single-valued relation, i.e. if $< t_1, t_2 > \in \delta$ and $< t_1, t_3 > \in \delta$, then $t_2 \equiv t_3$, where $t_1, t_2, t_3 \in \Lambda^T_M$;

2. For any constant term $f(t_1, \dots, t_k) \in \Lambda^T_M$ with $m \in M$ we have $f(t_1, \dots, t_k) \to\to_{\beta\delta} m$, where $f \in [M^k \to M]$, $t_1, \dots, t_k \in \Lambda^T_M$.

A natural notion of δ-reduction is called effective, natural notion of δ-reduction, if there exists an algorithm, which for any term $f(t_1, \dots, t_k)$, where $f \in [M^k \to M]$, $t_i \in \Lambda^T_M$, $i = 1, \dots, k$, $k \geq 1$, gives its convolution, if $f(t_1, \dots, t_k)$ is a δ-redex and stops with a negative answer otherwise. We will consider an effective, natural notion of δ-reduction such that every term has a single normal form, i.e. if $t \to\to_{\beta\delta} \tau_1$, $t \to\to_{\beta\delta} \tau_2$ and $\tau_1, \tau_2 \in NF^T \Rightarrow \tau_1 \equiv \tau_2$. The necessary and sufficient conditions for that are considered in [2].

Lemma 1.1. Let $t \in \Lambda^T$ and $t \to\to_{\beta\delta} \tau$, where $\tau \in NF^T$, then there exists a term t_0 such that $t \to\to_\beta t_0 \to\to_\delta \tau$.

Proof. Directly follows from the uniqueness of the normal form. □

Lemma 1.2. Let $t \in \Lambda^T$, $x \in V^T_\alpha, \alpha \in Types$ and $t \to\to_{\beta\delta} t_0 \in NF^T$, $FV(t_0) = \emptyset$, then for any $\tau \in \Lambda^T_\alpha$ we have: $t[x := \tau] \to\to_{\beta\delta} t_0$.

Proof. $\lambda x[t](\tau) \to\to_{\beta\delta} \lambda x[t_0](\tau) \to_\beta t_0$, $\lambda x[t](\tau) \to_\beta t[x := \tau]$ and the Lemma 1.2 follows from the uniqueness of the normal form. □

2. Untyped λ-terms. We fix a countable set of variables V. The set of terms is defined as follows:

- if $x \in V$, then $x \in \Lambda$;
- if $t_1, t_2 \in \Lambda$ then $(t_1 t_2) \in \Lambda$. The term $(t_1 t_2)$ is said to be obtained by the operation of application;
- if $x \in V$ then $t \in \Lambda$, then $(\lambda xt) \in \Lambda$. The term (λxt) is said to be obtained by the operation of abstraction.

The following shorthand notations are introduced: a term $(\ldots(t_1 t_2)\ldots t_k)$, where $t_i \in \Lambda, i = 1,\ldots,k,\ k > 1$, is denoted by $t_1 t_2 \ldots t_k$ and a term $(\lambda x_1(\lambda x_2(\ldots(\lambda x_n t)\ldots)$, where $x_j \in V,\ \ t \in \Lambda$, is denoted by $\lambda x_1 x_2 \ldots x_n.t,\ j = 1,\ldots,n,\ n > 0$.

The notions of free and bound occurrences of variables in untyped terms as well as the notion of free variable are introduced in the conventional way. The set of all free variables of an untyped term t is denoted by $FV(t)$. A term, which doesn't contain free variables, is called a closed term. Untyped terms t_1 and t_2 are said to be congruent ($t_1 \equiv t_2$), if one term can be obtained from the other by renaming bound variables. Congruent terms are considered identical.

We denote by $t[x_1 := t_1,\ldots,x_n := t_n]$ a term obtained by simultaneous substitution of terms $t_1,\ldots,t_n$ in the term t for all free occurrences of variables $x_1,\ldots,x_n$ respectively. A substitution is said to be admissible, if all free variables of substituted term remain free after the substitution. We consider only admissible substitutions.

A term t with a fixed occurrence of a subterm τ_1 is denoted by t_{τ_1}, and a term with this occurrence of τ_1 replaced by a term τ_2 is denoted by t_{τ_2}.

A term of the form $(\lambda x.t)\tau$ is called a β-redex, and the term $t[x := \tau]$ is called its convolution. A term t_1 is said to be obtained from a term t_0 by one-step β-reduction (denoted by $t_0 \to_\beta t_1$), if $t_0 \equiv t_{\tau_0}, t_1 \equiv t_{\tau_1}, \tau_0$ is a β-redex and τ_1 is its convolution. A term t is said to be obtained from a term t_0 by β-reduction (denoted by $t_0 \twoheadrightarrow_\beta t$), if there exists a finite sequence of terms $t_1,\ldots,t_n$ $(n \geq 1)$ such that $t_1 \equiv t_0,\ t_n \equiv t$ and $t_i \to_\beta t_{i+1}$, where $i = 1,\ldots,n-1$. A term containing no β-redexes is called a normal form. The set of all normal forms is denoted by NF. A term t is said to have a normal form, if there exists a term τ such that $\tau \in NF$ and $t \twoheadrightarrow_\beta \tau$.

From the Church–Rosser theorem it follows, that if $t \twoheadrightarrow_\beta \tau_1, t \twoheadrightarrow_\beta \tau_2$, $\tau_1, \tau_2 \in NF$, then $\tau_1 \equiv \tau_2$.

A term is said to be a head normal form, if it has a form $\lambda x_1 \ldots x_k.xt_1 \ldots t_n$, where $k,n \geq 0,\ t_1,\ldots,t_n \in \Lambda$. The set of all head normal forms is denoted by HNF. A term t is said to have a head normal form, if there exists a term τ such that $\tau \in HNF$ and $t \twoheadrightarrow_\beta \tau$. It is known, that $NF \subset HNF$, but $HNF \not\subset NF$ (see [3]).

Recall, that if a term has a head normal form, then the left reducing chain, where always the leftmost redex is chosen, leads to a head normal form, and if the term has a normal form, such reducing chain leads to the normal form (see [3]).

L e m m a 2 . 1 . Let t_b be a term with a fixed occurrence of a term b, which doesn't have a head normal form, and let c be any term, then:

1. $t_b \twoheadrightarrow_\beta \tau$, where $\tau \in NF \Rightarrow t_c \twoheadrightarrow_\beta \tau$;
2. t_b has a head normal form $\Rightarrow t_c$ has a head normal form.

Proof. It is easy to see, that Point 1 of Lemma 2.1 follows from the following statement: if $t_b \twoheadrightarrow_\beta \tau$, where $\tau \in NF$ and the left reducing chain's length is $k > 0$, then $t_c \twoheadrightarrow_\beta \tau$ and the left reducing chain's length is also k. We prove the statement. Obviously, the leftmost redex doesn't belong to the term b. Let the first leftmost redex have the form $(\lambda x.t_1)t_2$. We prove by induction on the reducing chain's length k. If $k = 1$ it is clear, that $x \notin FV(t_1)$ and the occurrence of b belongs to the subterm t_2, which proves the statement for the induction base. Let $k > 1$, we suppose that the statement holds for $k-1$ and prove it for k. There are 3 cases:

a) the occurrence of b belongs to the subterm t_2. Let t_1 have $n \geq 0$ free occurrences of the variable x, each occurrence of x in t_1 corresponds to an occurrence of b in the term obtained by redex convoluting. Sequentially replacing these n occurrences of b to c, we note that the terms obtained after each replacement reduce to τ and by the induction hypothesis the reducing chain's length is $k-1$. Also note, that the term obtained after these replacements can be also obtained after convoluting the leftmost redex in $t_c \Rightarrow t_c \twoheadrightarrow_\beta \tau$;

b) the occurrence of b belongs to the subterm t_1. Note that the term obtained by redex convoluting has an occurrence of the term $b[x := t_2]$, which doesn't have a head normal form as well (see [3]), reduces to τ and the reducing chain's length is $k-1$. By the induction hypothesis the term with this occurrence of $b[x := t_2]$ replaced by $c[x := t_2]$ reduces to τ and the reducing chain's length is $k-1$. Also note, that the term obtained after this replacement can be also obtained after convoluting the leftmost redex in $t_c \Rightarrow t_c \twoheadrightarrow_\beta \tau$;

c) the occurrence of b does not belong to the subterms t_1 and t_2. The statement follows from the induction hypothesis. The proof of Point 2 is similar to the proof of Point 1. $\square$

Lemma 2.2. Let $t \in \Lambda$ and $x \in V$, then we have the following: $t \twoheadrightarrow_\beta t_0$, where $t_0 \in NF$ and $FV(t_0) = \emptyset \Rightarrow$ for any term τ we have: $t[x := \tau] \twoheadrightarrow_\beta t_0$.

Proof. $(\lambda x.t)\tau \twoheadrightarrow_{\beta\delta} (\lambda x.t_0)\tau \rightarrow_\beta t_0, (\lambda x.t)\tau \rightarrow_\beta t[x := \tau]$ and Lemma 2.2 follows from the uniqueness of the normal form. $\square$

3. Translation. Let M be a recursive, countable, partially ordered set, which has a least element $\perp$ and every element of M is comparable with itself and with $\perp$. Function $f : M^k \to M$ $(k \geq 0)$ is called strong computable, if there exists an algorithm, which for any $m_1, \ldots, m_k \in M$ stops with the value $f(m_1, \ldots, m_k)$. Every $m \in M$ is mapped to an untyped term in the following way:

- $m \in M \setminus \{\perp\} \Rightarrow m' \in NF,\ FV(m') = \emptyset$ and for any $m_1, m_2 \in M \setminus \{\perp\}$, $m_1 \neq m_2 \Rightarrow m_1{}' \not\equiv m_2{}'$;
- $m \equiv \perp \Rightarrow m' \equiv \Omega \equiv (\lambda x.xx)(\lambda x.xx)$.

We say that an untyped term Φ λ-defines (see [4]) the function $f : M^k \to M$ $(k \geq 0)$, if for any $m_1, \ldots, m_k \in M$ we have the following:

$$f(m_1, \ldots, m_k) = m_0 \neq \perp \Rightarrow \Phi m_1{}' \ldots m_k{}' \twoheadrightarrow_\beta m_0{}',$$

$$f(m_1, \ldots, m_k) = \perp \Rightarrow \Phi m_1{}' \ldots m_k{}' \text{ does not have a head normal form.}$$

We consider typed terms using a set of functions C_1 such that all functions in C_1 are strong computable and for each $f \in C_1$ there exists an untyped term, which

λ-defines the function f. We assume that for the set C_1 there exists an effective, natural notion of δ-reduction such that every typed term has a single normal form.

We present an algorithm of translation of any typed term t to an untyped term t':

- $t \equiv m \in M \Rightarrow t' \equiv m'$;
- $t \in C_1 \Rightarrow FV(t') = \emptyset$ and t' λ-defines t;
- $t \equiv x \in V^T \Rightarrow x' \in V$ and $\forall x_1, x_2 \in V^T$, $x_1 \not\equiv x_2 \Rightarrow x_1{}' \not\equiv x_2{}'$;
- $t \equiv \tau(t_1, \ldots, t_n)$, $n \geq 1 \Rightarrow t' \equiv \tau' t_1{}' \ldots t_n{}'$;
- $t \equiv \lambda x_1 \ldots x_n[\tau]$, $n \geq 1 \Rightarrow t' \equiv \lambda x_1{}' \ldots x_n{}'.\tau'$.

Lemma 3.1. Let $t, \tau \in \Lambda^T$ and $t \twoheadrightarrow_\beta \tau$, then $t' \twoheadrightarrow_\beta \tau'$.

Proof. There exist typed terms $t_0, \ldots, t_k$ $(k \geq 0)$ such that $t \equiv t_0 \rightarrow_\beta t_1 \rightarrow_\beta \ldots \rightarrow_\beta t_k \equiv \tau$. The proof is by induction on the reducing chain's length $k \geq 0$. Lemma 3.1 is obvious for the basis of the induction, i.e. if $k = 0$. Let $k > 0$, we suppose that Lemma 3.1 holds for $k-1$. It is obvious that there exists a β-redex τ_0 such that τ_0 is a subterm of the term t, $\tau_0 \equiv \lambda x_1 \ldots x_n[a](b_1, \ldots, b_n), n \geq 1$, the term $\tau_1 \equiv a[x_1 := b_1, \ldots, x_n := b_n]$ is the convolution of τ_0 and $t_1 \equiv t_{\tau_1}$. Let $FV(b_i) \cap \{x_1, \ldots, x_n\} = \emptyset, i = 1, \ldots, n$, we can achieve this by renaming bound variables otherwise. It is easy to see, that $\tau_0{}' \equiv (\lambda x_1{}' \ldots x_n{}'.a')b_1{}' \ldots b_n{}'$ and $\tau_0{}' \twoheadrightarrow_\beta \tau_1{}'$, where $\tau_1{}' \equiv a'[x_1{}' := b_1{}', x_2{}' := b_2{}', \ldots, x_n{}' := b_n{}']$. It is also easy to see, that the term $\tau_0{}'$ has an occurrence in the term t', so replacing this occurrence with $\tau_1{}'$, we get the term $t_1{}'$ and therefore $t' \twoheadrightarrow_\beta t_1{}'$. Since t_1 reduces to τ and the reducing chain's length is $k-1$, by induction hypothesis $t_1{}' \twoheadrightarrow_\beta \tau'$ and, therefore, $t' \twoheadrightarrow_\beta \tau'$. □

Lemma 3.2. Let $t \in \Lambda_M^T, t \in \beta - NF, FV(t) = \emptyset, FV(\tau) = \emptyset, \tau \in NF$, then

1. $t \twoheadrightarrow_\delta m \in M \setminus \{\perp\}$ and $\tau \equiv m' \Leftrightarrow t' \twoheadrightarrow_\beta \tau$;
2. $t \twoheadrightarrow_\delta \perp \Leftrightarrow t'$ doesn't have a head normal form.

Proof. As $t \in \beta - NF$ and $FV(t) = \emptyset$, there exists a reducing chain such that every time a δ-redex is chosen of the form $f(m_1, \ldots, m_n)$, $m_i \in M$, $i = 1, \ldots, n$. We consider such chain. Let $t \equiv t_0 \rightarrow_\delta t_1 \rightarrow_\delta \ldots \rightarrow_\delta t_k \equiv m$, $t_i \in \Lambda_M^T$, $i = 1, \ldots, k$, $k \geq 0$. The proof is by induction on the reducing chain's length $k \geq 0$. Lemma 3.2($\Rightarrow$) is obvious for the basis of the induction, i.e. if $k = 0$. Let $k > 0$, we suppose, that Lemma 3.2($\Rightarrow$) holds for $k-1$. It is obvious that there exists a δ-redex τ_0 such that τ_0 is a subterm of the term t, the term τ_1 is the convolution of τ_0 and $t_1 \equiv t_{\tau_1}$. It is easy to see, that $\tau_1 \equiv m_0 \in M$. It is also easy to see, that the term $\tau_0{}'$ has an occurrence in the term t', and if we replace this occurrence with $m_0{}'$, then we get the term $t_1{}'$. If $m_0 \neq \perp$, then $f' m_1{}' \ldots m_n{}' \twoheadrightarrow_\beta m_0{}'$. If $m_0 = \perp$, then $f' m_1{}' \ldots m_n{}'$ does not have a head normal form and from Lemma 2.1 it follows, that if we replace the occurrence of $m_0{}' \equiv \Omega$ in $t_1{}'$ with $f' m_1{}' \ldots m_n{}'$, then either the resulting term will reduce to the same normal form as $t_1{}'$ or both of the terms will not have a head normal form. The term t_1 reduces to m or $\perp$ and the reducing chain's length is $k-1$, therefore, Lemma 3.2($\Rightarrow$) follows from the induction hypothesis. Assuming the opposite it is easy to see that Lemma 3.2($\Leftarrow$) directly follows from Lemma 3.2($\Rightarrow$). □

Theorem 3.1. Let $t \in \Lambda_M^T$, $FV(t) = \emptyset$, $\tau \in NF$ and $FV(\tau) = \emptyset$, then

1. $t \twoheadrightarrow_{\beta\delta} m, m \in M \setminus \{\perp\}$ and $\tau \equiv m' \Leftrightarrow t' \twoheadrightarrow_{\beta} \tau$;

2. $t \twoheadrightarrow_{\beta\delta} \perp \Leftrightarrow t'$ doesn't have a head normal form.

Proof. Theorem 3.1($\Rightarrow$) directly follows from Lemmas 1.1, 3.1 and 3.2. Assuming the opposite it is easy to see that Theorem 3.1($\Leftarrow$) follows from Theorem 3.1($\Rightarrow$). □

Theorem 3.2.

1. Let $t \in \Lambda_M^T, \tau \in NF$ and $FV(\tau) = \emptyset$, then $t \twoheadrightarrow_{\beta\delta} m,\ m \in M \setminus \{\perp\}$ and $\tau \equiv m' \Leftrightarrow t' \twoheadrightarrow_{\beta} \tau$.

2. There exists a term $t \in \Lambda_M^T$ such that $t \twoheadrightarrow_{\beta\delta} \perp$, but t' has a head normal form.

Proof.

Point 1. ($\Rightarrow$) We replace all free occurrences of variables in the term t with the terms that correspond to the least elements of the according types, obtained by the operation of abstraction and the term $\perp$. According to Lemma 1.2, the resulting term reduces to m, and, according to Theorem 3.1, the corresponding untyped term reduces to m'. Then, according to Lemma 2.1, the term t' also reduces to m'.

($\Leftarrow$) According to Lemma 1.1, there exists $t_0 \in \beta - NF^T$, such that $t \twoheadrightarrow_{\beta} t_0$. It follows from Lemma 3.1 that $t' \twoheadrightarrow_{\beta} t_0'$. Let $FV(t_0) = \{x_1, x_2, \dots, x_n\}, x_i \in V_{\alpha_i}^T$, $\alpha_i \in Types, n \geq 0$ and Ω_i be a term that represents the least element of α_i, obtained by the operation of abstraction and the term $\perp, i = 1, \dots, n$. It is easy to see, that $(t_0[x_1 := \Omega_1, \dots, x_n := \Omega_n])' \equiv t_0'[x_1' := \Omega_1', \dots, x_n' := \Omega_n']$. According to Lemma 2.2, $t_0'[x_1' := \Omega_1', \dots, x_n' := \Omega_n'] \twoheadrightarrow_{\beta} \tau$. Let $t_0[x_1 := \Omega_1, \dots, x_n := \Omega_n] \twoheadrightarrow_{\beta\delta} m$, it follows from Theorem 3.1 that $m \neq \perp$ and $m' \equiv \tau$. Suppose that $\Omega_1^0, \dots, \Omega_n^0$ are the least elements of the types $\alpha_1, \dots, \alpha_n$. Since $\Omega_i \sim \Omega_i^0 (1 \leq i \leq n)$ and $t_0[x_1 := \Omega_1, \dots, x_n := \Omega_n] \twoheadrightarrow_{\beta\delta} m$, it can be shown, that $Val_{<\Omega_1^0, \dots, \Omega_n^0>}(t_0) = m$. For any $a_1, a_2, \dots, a_n$, where $a_i \in \alpha_i$, $i = 1, \dots, n$, we have the following: $\langle \Omega_1^0, \dots, \Omega_n^0 \rangle \subseteq \langle a_1, \dots, a_n \rangle$ and $Val_{<a_1 \dots a_n>}(t_0) = m$. Consequently, t_0 is a constant term with m value. Since $t_0 \in \beta - NF$, either $t_0 \in M$, i.e $t_0 \equiv m$, or $t_0 \equiv f(t_1, \dots, t_k)$, $k \geq 0,\ f \in C_1,\ t_i \in \Lambda_M^T\ (0 \leq i \leq k)$, and according to the feature of a natural notion of δ-reduction we have: $f(t_1, \dots, t_k) \twoheadrightarrow_{\delta} m$.

To prove Point 2 of the Theorem, we give an example. Let $M = N \cup \{\perp\}$. If $n \in N$, then $n' \equiv \langle n \rangle$, where $\langle 0 \rangle \equiv \lambda x.x$ and $\langle n+1 \rangle \equiv \lambda x.xF\langle n \rangle, F \equiv \lambda xy.y$. Let $C_1 = \{f\}$, where for any $m \in M\ f(m) = \perp$. Let $f' \equiv \lambda x.(Zero\ x)\Omega\Omega$, where $Zero \equiv \lambda x.x(\lambda xyz.z)TFT, T \equiv \lambda xy.x$. Let $v \in V^T$, then it is clear, that $f(v) \twoheadrightarrow_{\beta\delta} \perp$, but the term $f'v'$ has a head normal form. □

Theorem 3.3.

1. For any terms $t_1, t_2 \in \Lambda^T$, such that $t_1 \twoheadrightarrow_{\beta\delta} t_2$, and there exists a reducing chain from the term t_1 to the term t_2, which is always choosing a δ-redex, the convolution of which is some $m \in M \setminus \{\perp\}$, we have the following: $t_1' \twoheadrightarrow_{\beta} t_2'$.

2. There exist terms $t_1, t_2 \in \Lambda^T$ such that $t_1 \twoheadrightarrow_{\beta\delta} t_2$, but t_1' does not reduce to t_2'.

Proof. We prove Point 1 by induction on the length $k \geq 0$ of the reducing chain from t_1 to t_2. Point 1 is obvious for the basis of the induction.

Suppose $k > 0$ and the statement holds for $k-1$. Let τ_1 be the first convoluted redex, τ_2 be its convolution and $t_1 \equiv t_{\tau_1}$. If τ_1 is a β-redex, then it follows from Lemma 3.1 that $t_1{}' \equiv t'_{\tau'_1} \rightarrow\!\!\rightarrow_\beta t'_{\tau_2{}'}$ and Point 1 follows from the induction hypothesis. If τ_1 is a δ-redex, then $\tau_2 \in M \setminus \{\perp\}$ and it follows from Theorem 3.2, that $t_1{}' \equiv t'_{\tau_1{}'} \rightarrow\!\!\rightarrow_\beta t'_{\tau_2{}'}$, and Point 1 also follows from the induction hypothesis. Point 2 of Theorem 3.3 directly follows from the Point 2 of Theorem 3.2. $\square$

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