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ON INTERVAL TOTAL COLORINGS OF BLOCK GRAPHS

N. A. KHACHATRYAN *

Chair of Discrete Mathematics and Theoretical Informatics YSU, Armenia

A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges get the same color. An interval total t-coloring of a graph G is a total coloring of G with colors $1,2,\dots,t$ such that all colors are used and the edges incident to each vertex v together with v are colored by $d_G(v)+1$ consecutive colors, where $d_G(v)$ is the degree of a vertex v in G. A block graph is a graph, in which every 2-connected component is a clique. In this paper we prove that all block graphs are interval total colorable. We also obtain some bounds for the smallest and greatest possible number of colors in interval total colorings of such graphs.

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Introduction. All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G respectively. The degree of a vertex v in G is denoted by $d_G(v)$, the maximum degree of vertices in G by $\Delta(G)$ and the total chromatic number of G by $\chi''(G)$. A block graph is a graph, in which every 2-connected component is a clique. The terms and concepts that we do not define can be found in [1–3] .

A total coloring of a graph G is a coloring of its vertices and edges such that no adjacent vertices, edges, and no incident vertices and edges get the same color. For a total coloring α of a graph G and for any $v \in V(G)$ define the set $S[v,\alpha]$ (spectrum of a vertex v) as follows:

$$S[v,\alpha] \equiv \{\alpha(v)\} \bigcup \{\alpha(e) | e \text{ is incident to } v\}.$$

Interval total t-coloring [4] of a graph G is a total coloring α of G with colors $1,2,\dots,t$ such that all colors are used and for any $v \in V(G)$, $S[v,\alpha]$ is an interval of integers. A graph G is interval total colorable if it has an interval total t-coloring for some positive integer t. The set of all interval total colorable graphs is denoted by $\mathfrak{T}$.

* E-mail: xachnerses@gmail.com

For a graph $G \in \mathfrak{T}$, the smallest (the minimum span) and the greatest (the maximum span) values of t, for which G has an interval t-coloring, are denoted by $w_\tau(G)$ and $W_\tau(G)$ respectively. Clearly,

$$\chi''(G) \leq w_\tau(G) \leq W_\tau(G) \leq |V(G)| + |E(G)| \text{ for every graph } G \in \mathfrak{T}.$$

The concept of interval total coloring was introduced by Petrosyan [4]. In particular, in [4, 5] is proved that if $m+n+2-gcd(m,n) \leq t \leq m+n+1$, then the complete bipartite graph $K_{m,n}$ has an interval total t-coloring. Interval total colorings of complete graphs and hypercubes are investigated in [5], where the following two theorems were proved.

Theorem 1. For any $n \in N$, we have:

1) $K_n \in \mathfrak{T}$,

2) $w_\tau(K_n) = \begin{cases} n, & \text{if } n \text{ is odd}, \\ \frac{3}{2}n, & \text{if } n \text{ is even}, \end{cases}$

3) $W_\tau(K_n) = 2n-1$.

Theorem 2. For any $n \in N$, we have:

1) $Q_n \in \mathfrak{T}$,

2) $w_\tau(Q_n) = \chi''(Q_n) = \begin{cases} n+2, & \text{if } n \leq 2, \\ n+1, & \text{if } n \geq 3, \end{cases}$

3) $W_\tau(Q_n) \geq \frac{(n+1)(n+2)}{2}$,

4) if $w_\tau(Q_n) \leq t \leq \frac{(n+1)(n+2)}{2}$, then Q has an interval total t-coloring.

Remark. Moreover, by the proof of Theorem 1 we have that extremal colors in the colorings are appearing on vertices.

Later, Petrosyan and Torosyan [6] showed that if $w_\tau(G) \leq t \leq W_\tau(G)$ then the complete graph K_n has an interval total t-coloring. Interval total colorings of bipartite graphs was investigated in [7]. In particular, it was shown that all trees are interval total colorable [8]. On the other hand, in [7, 9] examples of bipartite graphs that have no interval total coloring were constructed. Recently, it was proved that $W_\tau(Q_n) = \frac{(n+1)(n+2)}{2}$ for the hypercube [10].

Unfortunately it is known only few results related to the problem of determination of the exact values of the minimum and the maximum span in the interval total colorings of graphs. The exact values of these parameters are known only for paths, cycles, trees [8, 9], wheels [11], as well as for complete and complete balanced bipartite graphs [4–6]. In some papers [10–15] lower and upper bounds were found for the minimum and the maximum span in the interval total colorings of certain graphs.

In the present paper we show that all block graphs are interval total colorable. Moreover, we also derive some bounds and exact values of the parameters $w_\tau(G)$, $W_\tau(G)$ for a block graph G.

Interval Total Coloring of Block Graphs. In this section we present an algorithm constructing an interval total coloring of a given block graph G and then we derive some bounds for parameters $w_\tau(G)$ and $W_\tau(G)$. Let G be a block graph with blocks $C_1, C_2, \ldots, C_n$. It is well-known that all blocks are cliques. Let $V(G) = V(C_1) \cup V(C_2) \cup \ldots \cup V(C_n)$ and $E(G) = E(C_1) \cup E(C_2) \cup \ldots \cup E(C_n)$. The sequence $C_{i_1}, C_{i_2}, \ldots, C_{i_n}$ of blocks of a graph G, where $C_{i_p} \neq C_{i_q}$, $1 \leq p \leq q \leq m$, is called a path of cliques $P(C_{i_1}, C_{i_m})$ of G, if $V(C_{i_j}) \cap V(C_{i_{j+1}}) \neq \emptyset, (1 \leq j \leq m-1)$. For a path of cliques $P(C_{i_1}, C_{i_m})$ of G, define $MP(C_{i_1}, C_{i_m})$ as follows:

$$MP(C_{i_1}, C_{i_m}) = \sum_{j=1}^{m-1} d_G\big(V(C_{i_j}) \cap V(C_{i_{j+1}})\big) \text{ and } LP(C_{i_1}, C_{i_m}) = m.$$

The path of cliques $P(C_{i_1}, C_{i_m},)$ for which $MP(C_{i_1}, C_{i_m})$ has the maximum value among all possible paths of cliques of G, is called a hardest path of cliques of G. According to Theorem 1, each clique C has an interval total coloring α with colors $1, 2, \ldots, 2 \mid V(C) \mid -1$ and $W_\tau(C) = 2 \mid V(C) \mid -1$. Without loss of generality we may assume that in a clique C we can choose vertices v and v', $v \neq v'$, such that $1 \in S[v, \alpha]$ and $\big(2 \mid V(C) \mid -1\big) \in S[v', \alpha]$. Let $P(C_{i_1}, C_{i_m})$ be a hardest path of cliques of G. Now we construct a total coloring of G as follows:

Step 1. First we color the vertices and edges of the clique C_{i_1} interval totally such that $\big(2 \mid V(C) \mid -1\big) \in S[V(C_{i_1}) \cap V(C_{i_2}), \beta_1]$, where β_1 is the aforementioned coloring of a clique C_{i_1}.

Step 2. Next we color the cliques $C_{i_2}, \ldots, C_{i_{m-1}}$ interval totally each clique C_{i_j} by colors $k_{i_j}, 1 + k_{i_j}, \ldots, 2 \mid V(C_{i_j}) \mid -2 + k_{i_j}$, where

$$k_{i_j} = \sum_{p=1}^{j-1} \big(2 \mid V(C_{i_j}) \mid -2 + l_G\big((V(C_{i_p}) \cap V(C_{i_{p+1}}))\big)\big) - 1$$

and $l_G(v)$ (v is a cut-vertex of $P(C_{i_1}, C_{i_m})$) is the number of edges, which are incident to v and does not belong to $P(C_{i_1}, C_{i_m})$, such that $2 \mid V(C_{i_j}) \mid -2 + k_{i_j} \in S[V(C_{i_j}) \cap V(C_{i_{j+1}}), \beta_j]$ and k_{i_j} is a color of $V(C_{i_{j-1}}) \cap V(C_{i_j})$ where β_j is the aforementioned coloring of a clique C_{i_j}. Then we recolor the vertex $V(C_{i_{j-1}}) \cap V(C_{i_j})$ by color $\beta_{j-1}\big(V(C_{i_{j-1}}) \cap V(C_{i_j})\big)$. If there are uncolored blocks different from C_{i_j}, which are incident to $V(C_{i_{j-1}}) \cap V(C_{i_j})$, then we color all these blocks according to Step 4 before the coloring of C_{i_j}.

Step 3. Finally we color the vertices and edges of the clique C_{i_m} interval totally with colors $k_{i_m}, 1 + k_{i_m}, \ldots, 2 \mid V(C_{i_m}) \mid -2 + k_{i_m}$, and recolor the vertex $V(C_{i_{m-1}}) \cap V(C_{i_m})$ by color $\beta_{m-1}\big(V(C_{i_{m-1}}) \cap V(C_{i_m})\big)$.

Step 4. If we have an uncolored block C_r, which is incident to some colored block C_s, then we take any interval total coloring σ of C_r such that $1 \in S[V(C_r) \cap V(C_s), \sigma]$ and shift all colors by $a - 1$, where a is the maximum color of the spectrum of $V(C_r) \cap V(C_s)$. Then we recolor the vertex $V(C_r) \cap V(C_s)$ with its previous color.

It is not difficult to see that the resulting coloring is an interval total $\left(2 \sum_{j=1}^{m} \mid V(C_{i_j}) \mid -2LP(C_{i_1}, C_{i_m}) + \sum_{j=1}^{m-1} l_G\big(V(C_{i_j}) \cap V(C_{i_{j+1}})\big) + 1\right)$-coloring of G.

Thus, $W_\tau(G) \geq 2\sum_{j=1}^{m} \mid V(C_{i_j}) \mid -2LP(C_{i_1},C_{i_m}) + \sum_{j=1}^{m-1} l_G\big(V(C_{i_j}) \cap V(C_{i_{j+1}})\big) + 1$ (see Fig. 1).

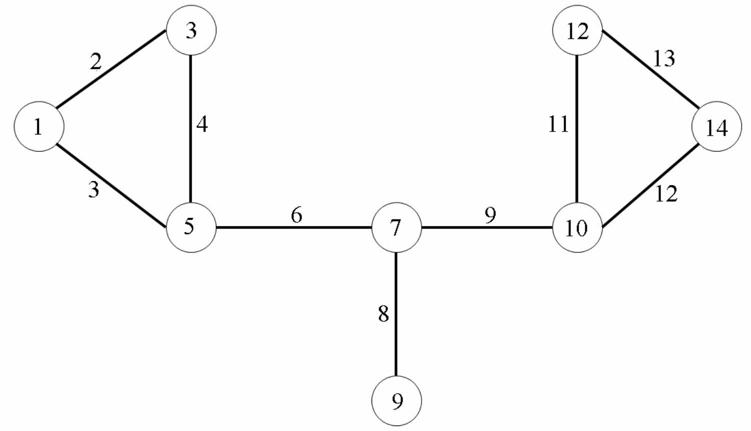

Fig. 1. An interval total coloring of given graph G.

Now we show that $W_\tau(G) \leq 2\sum_{j=1}^{m} \mid V(C_{i_j}) \mid -2LP(C_{i_1},C_{i_m}) + \sum_{j=1}^{m-1} l_G\big(V(C_{i_j}) \cap V(C_{i_{j+1}})\big) + 1$. Suppose, to the contrary, that γ is an interval total t-coloring of G, where $t > 2\sum_{j=1}^{m} \mid V(C_{i_j}) \mid -2LP(C_{i_1},C_{i_m}) + \sum_{j=1}^{m-1} l_G\big(V(C_{i_j}) \cap V(C_{i_{j+1}})\big) + 1$. Let C_{p_1} and C_{p_2} be blocks such that $1 \in \bigcup_{v\in V(C_{p_1})} S[v,\gamma]$ and $t \in \bigcup_{v\in V(C_{p_2})} S[v,\gamma]$ (it is possible that $p_1 = p_2$). Let $P'(C_{q_1},C_{q_s})$ be a path of cliques of G, where $C_{q_1} = C_{p_1}$ and $C_{q_s} = C_{p_2}$ (in the case $q_1 = q_s$, we take $C_{q_1} = C_{q_s} = P'(C_{q_1},C_{q_s})$). It is not difficult to see that

$$t \leq 2\sum_{j=1}^{s} \mid V(C_{q_j}) \mid -2LP'(C_{q_1},C_{q_s}) + \sum_{j=1}^{s-1} l_G\big(V(C_{q_j}) \cap V(C_{q_{j+1}})\big) + 1 \leq$$

$$\leq 2\sum_{j=1}^{m} \mid V(C_{i_j}) \mid -2LP(C_{i_1},C_{i_m}) + \sum_{j=1}^{m-1} l_G\big(V(C_{i_j}) \cap V(C_{i_{j+1}})\big) + 1,$$

which is a contradiction. So we have

$$W_\tau(G) = 2\sum_{j=1}^{m} \mid V(C_{i_j}) \mid -2LP(C_{i_1},C_{i_m}) + \sum_{j=1}^{m-1} l_G\big(V(C_{i_j}) \cap V(C_{i_{j+1}})\big) + 1.$$

Now we obtain an upper bound for $w_\tau(G)$ of a block graph G. Let v be a cut-vertex of G. For a vertex v define $g(v)$ as follows:

$$g(v) = \max_{P_v(C_{i_p},C_{i_q}),\, v\in V(C_{i_p})} MP_v(C_{i_p},C_{i_q}).$$

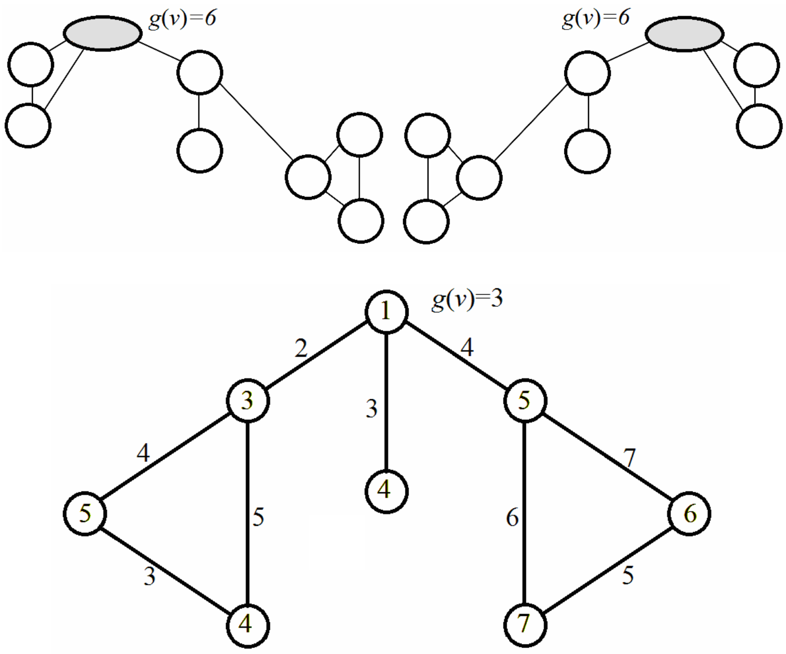

Fig. 2. Values of parameter $g(v)$ for a different cut-vertices of graph G shown in the Fig. 1.

Moreover, let $P_v^*(C_{i_p}^*, C_{i_q}^*)$ be a path, for which $g(v)$ is attained. Let v^* be a vertex such that $g(v^*) = \min\limits_{v \in V(G)} g(v)$ (Fig. 2).

By Theorem 1, we have $w_\tau(C_n) \le \frac{3}{2}n$. From this, taking into account Remark, by the same algorithm, we obtain that G has an interval total coloring by no more than $\frac{3}{2}\sum\limits_{j=p}^{q} \mid V(C_{i_j}^*) \mid - LP_{v^*}^*(C_{i_p}^*, C_{i_q}^*) + \sum\limits_{j=p}^{q-1} l_G\big(V(C_{i_j}^* \cap V(C_{i_{j+1}}^*)\big) + 1$ colors. Based on these results, we can formulate the following theorem.

Theorem 3. If G is a block graph, then

1) $G \in \mathfrak{T}$;

2) $W_\tau(G) = 2\sum\limits_{j=1}^{m} \mid V(C_{i_j}) \mid - 2LP(C_{i_1}, C_{i_m}) + \sum\limits_{j=1}^{m-1} l_G\big(V(C_{i_j}) \cap V(C_{i_{j+1}})\big) + 1$;

3) $w_\tau(G) \le \frac{3}{2}\sum\limits_{j=p}^{q} \mid V(C_{i_j}^*) \mid - LP_{v^*}^*(C_{i_p}^*, C_{i_q}^*) + \sum\limits_{j=p}^{q-1} l_G\big(V(C_{i_j}^* \cap V(C_{i_{j+1}}^*)\big) + 1$.

Our results generalize the results on interval total colorings of complete graphs [5,6] and trees [8], since all these graphs are block graphs.

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