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SOME RELATIONS BETWEEN THE μ-PARAMETERS OF REGULAR GRAPHS

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We consider undirected, simple, finite, connected graphs. Some relations between the μ-parameters are obtained for the case of regular graphs.

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Introduction. We consider finite, undirected, connected graphs without loops and multiple edges containing at least one edge. For any graph G we denote by $V(G)$ and $E(G)$ the sets of vertices and edges of G, respectively. For any $x \in V(G)$ $d_G(x)$ denotes the degree of the vertex x in G. For a graph G $\delta(G)$ and $\Delta(G)$ denote the minimum and the maximum degree of a vertex in G, respectively.

An arbitrary nonempty finite subset of consecutive integers is called an interval. An interval with the minimum element p and the maximum element q is denoted by $[p,q]$.

A function $\varphi : E(G) \to [1,t]$ is called a proper edge t-coloring of a graph G, if each of t colors is used, and adjacent edges are colored differently.

The minimum value of t for which there exists a proper edge t-coloring of a graph G, is denoted by $\chi'(G)$ [1].

For any graph G and for any $t \in [\chi'(G), |E(G)|]$ we denote by $\alpha(G,t)$ the set of all proper edge t-colorings of G.

Let us also define a set $\alpha(G)$ of all proper edge colorings of a graph G:

$$\alpha(G) \equiv \bigcup_{t=\chi'(G)}^{|E(G)|} \alpha(G,t).$$

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If $\varphi \in \alpha(G)$ and $x \in V(G)$, then the set $\{\varphi(e)/e \in E(G), e \text{ is incident with } x\}$ is called a spectrum of the vertex x of the graph G at the proper edge coloring φ and is denoted by $S_G(x,\varphi)$.

If G is a graph, $\varphi \in \alpha(G)$, then set

$$V_{int}(G,\varphi) \equiv \{x \in V(G)/S_G(x,\varphi) \text{ is an interval}\}$$

and

$$f_G(\varphi) \equiv |V_{int}(G,\varphi)|.$$

A proper edge coloring $\varphi \in \alpha(G)$ is called an interval edge coloring [2–4] of the graph G, if and only if $f_G(\varphi) = |V(G)|$. The set of all graphs having an interval edge coloring is denoted by $\mathfrak{N}$. The terms and concepts, which are not defined can be found in [5].

For a graph G and for any $t \in [\chi'(G), |E(G)|]$, we set [6]:

$$\mu_1(G,t) \equiv \min_{\varphi \in \alpha(G,t)} f_G(\varphi), \qquad \mu_2(G,t) \equiv \max_{\varphi \in \alpha(G,t)} f_G(\varphi).$$

For any graph G, we set [6]:

$$\mu_{11}(G) \equiv \min_{\chi'(G) \le t \le |E(G)|} \mu_1(G,t), \qquad \mu_{12}(G) \equiv \max_{\chi'(G) \le t \le |E(G)|} \mu_1(G,t),$$

$$\mu_{21}(G) \equiv \min_{\chi'(G) \le t \le |E(G)|} \mu_2(G,t), \qquad \mu_{22}(G) \equiv \max_{\chi'(G) \le t \le |E(G)|} \mu_2(G,t).$$

Clearly, the μ-parameters are correctly defined for an arbitrary graph. Some remarks on their interpretations in games are given in [7].

The exact values of the parameters μ_{11}, μ_{12}, μ_{21} and μ_{22} are found for simple paths, simple cycles and simple cycles with a chord [8, 9], “Möbius ladders” [6, 10], complete graphs [11], complete bipartite graphs [12, 13], prisms [10, 14], n-dimensional cubes [14, 15] and the Petersen graph [7]. The exact values of μ_{11} and μ_{22} for trees are found in [16]. The exact value of μ_{12} for an arbitrary tree is found in [17] (see also [18, 19]).

In this paper some relations between the μ-parameters of regular graphs are obtained.

The Main Results. In the rest part of this paper we admit an additional condition: an arbitrary graph G satisfies the inequality $\delta(G) \ge 2$.

Theorem 1. [8, 9]. For any integer $k \ge 2$ the following equalities hold:

1. $\mu_{12}(C_{2k}) = \mu_{22}(C_{2k}) = 2k$,

   $\mu_{21}(C_{2k}) = 2k - 1$;

2. $\mu_{11}(C_{2k}) = \begin{cases} 1, & \text{if } k = 2, \\ 0, & \text{if } k \ge 3. \end{cases}$

Theorem 2. [8,9]. For any positive integer k the following equalities hold:

1. $\mu_{12}(C_{2k+1}) = 2$,

$\mu_{21}(C_{2k+1}) = \mu_{22}(C_{2k+1}) = 2k$;

2. $\mu_{11}(C_{2k+1}) = \begin{cases} 2, & \text{if } k = 1, \\ 0, & \text{if } k \geq 2. \end{cases}$

Corollary 1. [8, 9]. For any integer $k \geq 2$ the inequalities $\mu_{21}(C_{2k}) < \mu_{12}(C_{2k})$ and $\mu_{12}(C_{2k+1}) < \mu_{21}(C_{2k+1})$ hold.

Theorem 3. [8,9]. For any graph G the inequalities $\mu_{11}(G) \leq \mu_{12}(G) \leq \leq \mu_{22}(G)$, $\mu_{11}(G) \leq \mu_{21}(G) \leq \mu_{22}(G)$ hold.

Corollary 1 means that there are graphs G, for which $\mu_{21}(G) < \mu_{12}(G)$, and there are also graphs G, for which $\mu_{12}(G) < \mu_{21}(G)$.

Theorem 4. [8]. If G is a regular graph with $\chi'(G) = \Delta(G)$, then $\mu_{12}(G) = |V(G)|$.

Theorem 5. [20]. If G is an r-regular graph and $\varphi \in \alpha(G, |E(G)|)$, then

$$|V_{int}(G, \varphi)| \leq \left\lfloor \frac{r \cdot |V(G)| - 2}{2 \cdot (r-1)} \right\rfloor.$$

Corollary 2. If G is an r-regular graph, then

$$\mu_2(G, |E(G)|) \leq \left\lfloor \frac{r \cdot |V(G)| - 2}{2 \cdot (r-1)} \right\rfloor.$$

Corollary 3. If G is an r-regular graph, then

$$\mu_{21}(G) \leq \left\lfloor \frac{r \cdot |V(G)| - 2}{2 \cdot (r-1)} \right\rfloor.$$

Proposition. For arbitrary integers $r \geq 2$ and $n \geq 1$ the inequality

$$\left\lfloor \frac{r \cdot n - 2}{2 \cdot (r-1)} \right\rfloor \leq n - 1$$

holds.

Proof.

$$\left\lfloor \frac{rn-2}{2 \cdot (r-1)} \right\rfloor = \left\lfloor \frac{n}{2} + \frac{n-2}{2 \cdot (r-1)} \right\rfloor \leq \left\lfloor \frac{n}{2} + \frac{n-2}{2} \right\rfloor = n - 1. \qquad \square$$

Corollary 4. If G is a regular graph, then $\mu_{21}(G) \leq |V(G)| - 1$.

From Corollary 4 and Theorem 4 we obtain:

Corollary 5. For an arbitrary regular graph G with $\chi'(G) = \Delta(G)$ the inequality $\mu_{21}(G) < \mu_{12}(G)$ holds.

Theorem 6. For an arbitrary regular graph G the following four statements are equivalent:

1. $\chi'(G) = \Delta(G)$, 2. $G \in \mathfrak{N}$,

3. $\mu_{22}(G) = |V(G)|$, 4. $\mu_{12}(G) = |V(G)|$.

Proof. The equivalence between 1) and 2) was proved in [2–4]. The equivalence between 2) and 3) is evident.

Let us show the equivalence between 1) and 4).

If $\chi'(G) = \Delta(G)$, then by Theorem 4 we have the equality $\mu_{12}(G) = |V(G)|$. It means that 1) $\Rightarrow$ 4).

Now suppose that $\mu_{12}(G) = |V(G)|$. By Theorem 3, we have also the equality $\mu_{22}(G) = |V(G)|$. Consequently, using the equivalence between 2) and 3), we have also the relation $G \in \mathfrak{N}$. Finally, using the equivalence between 1) and 2), we have also the equality $\chi'(G) = \Delta(G)$. Thus, 4) $\Rightarrow$ 1). $\square$

Theorem 6 implies that the problem of determined whether $\mu_{12}(G) = |V(G)|$ for a given regular graph G is NP-complete.

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