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Mathematics

NONSELFADJOINT DEGENERATE DIFFERENTIAL EQUATIONS OF HIGHER ORDER

L. P. TEPOYAN *

Chair of Differential Equations YSU, Armenia

In this article we consider Dirichlet problem for some class of degenerate nonselfadjoint differential equations of higher order. We prove the existence and uniqueness of the generalized solution, establish analogue of the Keldysh theorem and explore the spectral properties of the corresponding operator.

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1. Introduction. The main focus of the present paper is on the degenerate differential equation:

$$Lu \equiv (-1)^m \big(t^{\alpha} u^{(m)}\big)^{(m)} + a\big(t^{\alpha-1} u^{(m)}\big)^{(m-1)} + pt^{\beta} u = f(t), \tag{1}$$

where $m \in \mathbb{N}$, t belongs to the finite interval $(0,b)$, $\alpha \geq 0$, $\alpha \neq 1,3,\ldots,2m-1$, $\beta \geq \alpha - 2m$, $a \neq 0$, a and p are real constants, $f \in L_{2,-\beta}(0,b)$, i.e.

$$\|f\|^2_{L_{2,-\beta}(0,b)} = \int_0^b t^{-\beta} |f(t)|^2 \, dt < \infty.$$

The dependence of the character of the boundary conditions with respect to t for $t = 0$ from the "lower order" terms was first observed by M.V. Keldish in [1] for the degenerate second order elliptic equation in a region in the plane. The case $m = 1$, $\beta = 0$, $0 \leq \alpha < 2$ was considered in [2, 3] and the case $m = 2$, $\beta = 0$, $0 \leq \alpha \leq 4$ in [4]. In [5] the self-adjoint case of higher order degenerate differential equations with arbitrary weight function on a finite interval has been considered. Note also that here have been used the method suggested by A.A. Dezin (see [6]).

2. Weighted Sobolev Spaces $\dot{W}^m_{\alpha}(0,b)$.

Let $\dot{C}^m[0,b]$ denote the set of functions $u \in C^m[0,b]$, satisfing

$$u^{(k)}(0) = u^{(k)}(b) = 0, \quad k = 0,1,\ldots,m-1. \tag{2}$$

* E-mail: tepoyan@yahoo.com

Define $\dot{W}^m_\alpha(0,b)$ as the completion of $\dot{C}^m[0,b]$ in the norm

$$\|u\|^2_{\dot{W}^m_\alpha(0,b)} = \int_0^b t^\alpha |u^{(m)}(t)|^2\,dt.$$

Denote the corresponding scalar product in $\dot{W}^m_\alpha(0,b)$ by $\{u,v\}_\alpha = (t^\alpha u^{(m)}, v^{(m)})$, where $(\cdot,\cdot)$ stands for the scalar product in $L_2(0,b)$.

Observe that for every $t_0 \in (\varepsilon, b], \varepsilon > 0$, any function $u \in \dot{W}^m_\alpha(0,b)$ has finite derivatives $u^{(k)}(t_0), k = 0,1,\ldots,m-1$, and $u^{(k)}(b) = 0,\ k = 0,1,\ldots,m-1$ (see [7]).

The proofs of the following two Propositions can be found, for instance, in [8].

P r o p o s i t i o n 1. For any $u \in \dot{W}^m_\alpha(0,b), \alpha \neq 1,3,\ldots,2m-1$, we have the following estimates

$$|u^{(k)}(t)|^2 \le C_1 t^{2m-2k-1-\alpha} \|u\|^2_{\dot{W}^m_\alpha(0,b)}, \quad k = 0,1,\ldots,m-1. \tag{3}$$

It follows from Proposition 1 that in the case $\alpha < 1$ (weak degeneracy), $u^{(j)}(0) = 0$ for all $j = 0,1,\ldots,m-1$, while for $\alpha > 1$ (strong degeneracy) not all $u^{(j)}(0) = 0$. More precisely, for $1 < \alpha < 2m-1$ the derivatives $u^{(j)}(0) = 0$ only for $j = 0,1,\ldots,s_\alpha$, where $s_\alpha = m - 1 - \left[\dfrac{\alpha+1}{2}\right]$ (here $[a]$ is the integer part of a), and for $\alpha > 2m-1$ all $u^{(j)}(0),\ \ j = 0,1,\ldots,m-1$, in general, can be infinite.

Denote $L_{2,\beta}(0,b) = \left\{ f, \displaystyle\int_0^b t^\beta |f(t)|^2\,dt < +\infty \right\}$. Observe that for $\alpha \le \beta$ we have a continuous embedding $L_{2,\alpha}(0,b) \subset L_{2,\beta}(0,b)$.

P r o p o s i t i o n 2. For $\beta \ge \alpha - 2m$ we have a continuous embedding

$$\dot{W}^m_\alpha(0,b) \subset L_{2,\beta}(0,b), \tag{4}$$

which is compact for $\beta > \alpha - 2m$.

Note that the embedding (4) for $\beta = \alpha - 2m$ is not compact and fails in the case $\beta < \alpha - 2m$. Let $d(m,\alpha) = 4^{-m}(\alpha-1)^2(\alpha-3)^2\cdots(\alpha-(2m-1))^2$. In Proposition 2 (see [8]), using the inequality of Hardy (see [9]), it has been proved that

$$\int_0^b t^\alpha |u^{(m)}(t)|^2\,dt \ge d(m,\alpha) \int_0^b t^{\alpha-2m} |u(t)|^2\,dt, \tag{5}$$

where the constant $d(m,\alpha)$ is exact. It is easy to verify that for $\beta \ge \alpha - 2m$ we have

$$\|u\|^2_{\dot{W}^m_\alpha(0,b)} \ge b^{\alpha-2m-\beta} d(m,\alpha) \|u\|^2_{L_{2,\beta}(0,b)}. \tag{6}$$

3. Nonselfadjoint Degenerate Equations. In this section we investigate the equation (1) for $a > 0$. First we define the generalized solution of Dirichlet problem.

D e f i n i t i o n 1 . A function $u \in \dot{W}^m_\alpha(0,b)$ is called a generalized solution of equation (1), if for arbitrary $v \in \dot{W}^m_\alpha(0,b)$ the following equality holds:

$$\{u,v\}_\alpha + a(-1)^{m-1}\left(t^{\alpha-1}u^{(m)}, v^{(m-1)}\right) + p(t^\beta u, v) = (f,v). \tag{7}$$

T h e o r e m 1 . Assume that

$$\begin{gathered} a(\alpha-1)(-1)^m > 0, \\ \gamma = b^{\alpha-2m-\beta}\left(d(m,\alpha) + \frac{a}{2}(\alpha-1)(-1)^m d(m-1,\alpha-2)\right) + p > 0. \end{gathered} \tag{8}$$

Then the generalized solution of the equation (7) exists and is unique for every $f \in L_{2,-\beta}(0,b)$.

Proof.

Uniqueness. To prove the uniqueness of the solution, we take in (7) $f=0$ and $v=u$. Let $\alpha>1$. By integrating by parts, we get

$$\left(t^{\alpha-1}u^{(m)},u^{(m-1)}\right)=-\frac{1}{2}\left(t^{\alpha-1}|u^{(m-1)}(t)|^2\right)\Big|_{t=0}-\frac{\alpha-1}{2}\int_0^b t^{\alpha-2}|u^{(m-1)}(t)|^2\,dt.$$

It follows from (3), for $k=m-1$, that the value $\left(t^{\alpha-1}|u^{(m-1)}(t)|^2\right)\Big|_{t=0}$ is finite. On the other hand, using (5), we get

$$\int_0^b t^{\alpha-2}|u^{(m-1)}(t)|^2\,dt \ge d(m-1,\alpha-2)\int_0^b t^{\alpha-2m}|u(t)|^2\,dt.$$

Hence, using the inequality (6), we obtain

$$0=\{u,u\}_\alpha+a(-1)^{m-1}\left(t^{\alpha-1}u^{(m)},u^{(m-1)}\right)+p(t^\beta u,u)\ge$$
$$\ge\frac{a}{2}(-1)^m\left(t^{\alpha-1}|u^{(m-1)}(t)|^2\right)\Big|_{t=0}+\gamma\int_0^b t^\beta\left|u(t)\right|^2dt.$$

Now the uniqueness of the generalized solution follows from the condition (8).

Existence. To prove the existence of the generalized solution, we define a linear functional $l_f(v)=(f,v),v\in\dot{W}^m_\alpha(0,b)$. From the continuity of the embedding (4) it follows that

$$|l_f(v)|\le\|f\|_{L_{2,-\beta}(0,b)}\|v\|_{L_{2,\beta}(0,b)}\le c\|f\|_{L_{2,-\beta}(0,b)}\|v\|_{\dot{W}^m_\alpha(0,b)},$$

therefore, the linear functional $l_f(v)$ is bounded on $\dot{W}^m_\alpha(0,b)$. Hence, it can be represented in the form $l_f(v)=(f,v)=\{u^*,v\},u^*\in\dot{W}^m_\alpha(0,b)$ (this follows from the Riesz theorem on the representation of the linear continuous functional). The last two terms in the left hand-side of the equality (7) also can be regarded as a continuous linear functional with respect to u and represented in the form $\{u,Kv\}_\alpha,Kv\in\dot{W}^m_\alpha(0,b)$. In fact, using the inequality (5), we can write

$$|a(-1)^{m-1}\left(t^{\alpha-1}u^{(m)},v^{(m-1)}\right)+p(t^\beta u,v)|\le|a(t^{\frac{\alpha}{2}}u^{(m)},t^{\frac{\alpha}{2}-1}v^{(m-1)})|+|p\left(t^{\frac{\beta}{2}}u,t^{\frac{\beta}{2}}v\right)|\le$$
$$\le c_1\|u\|_{\dot{W}^m_\alpha(0,b)}\left\{\int_0^b t^{\alpha-2}|v^{(m-1)}(t)|^2\,dt\right\}^{1/2}+c_2\|u\|_{L_{2,\alpha-2m}(0,b)}\|v\|_{L_{2,\alpha-2m}(0,b)}\le$$
$$\le\frac{2c_1}{|\alpha-1|}\|u\|_{\dot{W}^m_\alpha(0,b)}\|v\|_{\dot{W}^m_\alpha(0,b)}+c_3\|u\|_{\dot{W}^m_\alpha(0,b)}\|v\|_{\dot{W}^m_\alpha(0,b)}=c\|u\|_{\dot{W}^m_\alpha(0,b)}\|v\|_{\dot{W}^m_\alpha(0,b)}.$$

From equality (7) we deduce that for any $v\in\dot{W}^m_\alpha(0,b)$ we have

$$\left\{u,(I+K)v\right\}_\alpha=\{u^*,v\}_\alpha. \tag{9}$$

Observe that the image of the operator $I+K$ is dense in $\dot{W}^m_\alpha(0,b)$. Indeed, if for some $u_0\in\dot{W}^m_\alpha(0,b)$

$$\Big\{u_0,(I+K)v\Big\}_\alpha=0$$

for every $v\in\dot{W}^m_\alpha(0,b)$, then $u_0=0$, since we have already proved the uniqueness of the generalized solution for (7).

Assume that $0 < \sigma d(m,\alpha) b^{\alpha-2m-\beta} \le \gamma$. Then we can write

$$\{u,(I+K)u\}_\alpha \ge \sigma\{u,u\}_\alpha + \big(b^{\alpha-2m-\beta}\big((1-\sigma)d(m,\alpha)+$$
$$+\frac{a}{2}(\alpha-1)(-1)^m d(m-1,\alpha-2)\big)+p\big)\int_0^b t^\beta |u(t)|^2\,dt =$$
$$=\sigma\{u,u\}_\alpha + \big(\gamma - \sigma d(m,\alpha) b^{\alpha-2m-\beta}\big)\int_0^b t^\beta |u(t)|^2\,dt \ge$$
$$\ge \sigma\{u,u\}_\alpha.$$

Finally, we get

$$\{u,(I+K)u\}_\alpha \ge \sigma\{u,u\}_\alpha. \tag{10}$$

From (10) it follows that $(I+K)^{-1}$ is defined on $\dot{W}^m_\alpha(0,b)$ and is bounded. Consequently, there exists the operator $I+K^*$ and $(I+K^*)^{-1} = ((I+K)^{-1})^*$ (here K^* denotes the adjoint operator). Hence, from (9) we obtain

$$u = (I+K^*)^{-1}u^*.$$

For $\alpha < 1$ the proof is similar, and we use the fact that $\big(t^{\alpha-1}|u^{(m-1)}(t)|^2\big)\Big|_{t=0} = 0$, which follows from Proposition 1. □

Now we define an operator $S : D(S) \subset \dot{W}^m_\alpha(0,b) \subset L_{2,\beta}(0,b) \to L_{2,-\beta}(0,b)$.

Definition 2. We say that $u \in \dot{W}^m_\alpha(0,b)$ belongs to $D(S)$, if there exists a function $f \in L_{2,-\beta}(0,b)$ satisfying to (7) for every $v \in \dot{W}^m_\alpha(0,b)$. In this case we write $Su = f$.

The operator S acts from the space $L_{2,\beta}(0,b)$ to $L_{2,-\beta}(0,b)$. It is easy to check that the operator $\mathbb{S} := t^{-\beta}S, D(\mathbb{S}) = D(S)$, $\mathbb{S} : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ is an operator in the space $L_{2,\beta}(0,b)$, since if $f \in L_{2,-\beta}(0,b)$, then $f_1 := t^{-\beta} f \in L_{2,\beta}(0,b)$ and $\|f\|_{L_{2,-\beta}(0,b)} = \|f_1\|_{L_{2,\beta}(0,b)}$.

Proposition 3. Under the assumptions of Theorem 1, the inverse operator $\mathbb{S}^{-1} : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ is continuous for $\beta \ge \alpha - 2m$ and is compact for $\beta > \alpha - 2m$.

Proof. For the proof first observe that for $u \in D(\mathbb{S})$ we have

$$\|u\|_{L_{2,\beta}(0,b)} \le c\|f\|_{L_{2,-\beta}(0,b)} = c\|f_1\|_{L_{2,\beta}(0,b)}.$$

In fact, by setting $v = u$ in the equality (7), and by using the inequalities (6), (10), and applying considerations of Theorem 1 proof, we get

$$\sigma b^{\alpha-2m-\beta} d(m,\alpha)\|u\|^2_{L_{2,\beta}(0,b)} \le \sigma d(m,\alpha)\|u\|^2_{\dot{W}^m_\alpha(0,b)} \le$$
$$\le \{(I+K)u,u\}_\alpha = (f,u) \le \|f\|_{L_{2,-\beta}(0,b)}\|u\|_{L_{2,\beta}(0,b)} = \|f_1\|_{L_{2,\beta}(0,b)}\|u\|_{L_{2,\beta}(0,b)}.$$

Thus, we obtain

$$\|\mathbb{S}^{-1} f_1\|_{L_{2,\beta}(0,b)} \le c\|f_1\|_{L_{2,\beta}(0,b)}, \tag{11}$$

consequently, the continuity of $\mathbb{S}^{-1}$ for $\beta \ge \alpha - 2m$ is proved.

To show the compactness of $\mathbb{S}^{-1}$ for $\beta < \alpha - 2m$ it is enough to apply the compactness of the embedding (4) for $\beta < \alpha - 2m$. □

Let us consider the following equation

$$Tv \equiv (-1)^m \big(t^\alpha v^{(m)}\big)^{(m)} - a\big(t^{\alpha-1} v^{(m-1)}\big)^{(m)} + pt^\beta v = g(t), \tag{12}$$

where $\alpha \geq 0, \alpha \neq 1, 3, \dots, 2m-1$, $\beta \geq \alpha - 2m$, $g \in L_{2,-\beta}(0,b)$, $a \neq 0$, a and p are real constants.

Definition 3. We say that $v \in L_{2,\beta}(0,b)$ is a generalized solution of the equation (12), if for every $u \in D(S)$ the following equality holds:

$$(Su, v) = (u, g). \tag{13}$$

Let $g_1 := t^{-\beta} g$. Definition 3 of the generalized solution as above defines an operator $\mathbb{T} : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$, $\mathbb{T} := t^{-\beta} T$. Actually, we have defined the operator $\mathbb{T}$ as the adjoint to $\mathbb{S}$ operator in $L_{2,\beta}(0,b)$, i.e.

$$\mathbb{T} = \mathbb{S}^*.$$

Theorem 2. Under the assumptions of Theorem 1, the generalized solution of the equation (12) exists and is unique for every $g \in L_{2,-\beta}(0,b)$. Moreover, the inverse operator $\mathbb{T}^{-1} : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ is continuous for $\beta \geq \alpha - 2m$ and is compact for $\beta > \alpha - 2m$.

Proof. Solvability of the equation $\mathbb{S}u = f_1$ for any $f_1 \in L_{2,-\beta}(0,b)$ (see Theorem 1) implies the uniqueness of the solution of equation (12), while existence of the bounded inverse operator $\mathbb{S}^{-1}$ (see Proposition 3) implies the solvability of (12) for any $g \in L_{2,-\beta}(0,b)$ (see, for instance, [10]). Since we have $(\mathbb{S}^*)^{-1} = (\mathbb{S}^{-1})^*$, boundedness and compactness of the operator $\mathbb{S}^{-1}$ imply boundedness and compactness of the operator $\mathbb{T}^{-1}$ for $\beta \geq \alpha - 2m$, and $\beta > \alpha - 2m$ respectively (see Proposition 3). □

At the end we establish the spectral properties of the operators $\mathbb{S}$ and $\mathbb{T}$.

Proposition 4. The spectra of the operators $\mathbb{S}$ and $\mathbb{T}$ are in the right half-plane.

Proof. Since the operator $\mathbb{T}$ is adjoint to $\mathbb{S}$, therefore, it is sufficient to prove the statement for the operator $\mathbb{S}$. It follows from Proposition 3, that $0 \in \rho(\mathbb{S})$, where $\rho(\mathbb{S})$ means the resolvent set of the operator $\mathbb{S}$. Second inequality in (8) will be evidently satisfied, if we put instead of the number p some number $p_1 > p$. Let $\operatorname{Re}\lambda < 0$. Then, similar to Proposition 3, we can prove that for any $f \in L_{2,\beta}(0,b)$ (see also [2])

$$\|(\mathbb{S} - \lambda I)^{-1} f_1\|_{L_{2,\beta}(0,b)} \leq c \|f_1\|_{L_{2,\beta}(0,b)},$$

where I is the identity operator in $L_{2,\beta}(0,b)$. □

Remark 1. For $\alpha > 1$ and for every generalized solution v of equation (12) we have

$$\left(t^{\alpha-1} |u^{(m-1)}(t)|^2\right)\Big|_{t=0} = 0. \tag{14}$$

In fact, by replacing g by Tv in (13), and then integrating by parts the second term and using equality (7), we obtain (14). Note also that for the equation (7) the left-hand side of (14) is merely bounded. This is an analogue of the Keldish theorem (see [1]).

R e m a r k 2 . Another interesting phenomenon connected with degenerate equations is the appearance of the continuous spectrum. Assume that in Eq. (7) $a = p = 0$ and $\beta = \alpha - 2m$. In [8] it has been proved that the spectrum of the operator

$$Bu := (-1)^m t^{2m-\alpha} \left(t^\alpha u^{(m)}\right)^{(m)}, \quad B : L_{2,\alpha-2m}(0,b) \to L_{2,\alpha-2m}(0,b)$$

is purely continuous and coincides with the ray $[d(m,\alpha),+\infty)$. Note also that the spectrum of the operator $Qu := (-1)^m t^{-\beta}(t^\alpha u^{(m)})^{(m)}$, $Q : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ for $\beta > \alpha - 2m$ is discrete.

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