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Mathematics

ON FOURIER COEFFICIENTS WITH RESPECT TO THE WALSH DOUBLE SYSTEM

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In the present paper we will consider the behavior of Fourier coefficients with respect to the Walsh double system after modification of functions. We prove that for any function $f(x,y) \in L^p[0,1]^2$ one can find a function $g \in L^p[0,1]^2$ coinciding with $f(x,y)$ except a set of small measure such that the non-zero coefficients of $g(x,y)$ are monotonically decreasing over all rays in absolute value.

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1. Introduction. We will consider the behavior of Fourier coefficients with respect to the Walsh double system after modification of functions. Note that the well-known classical theorems of N.N. Lusin [1] and D.E. Men'shov [2] are about the "correction of functions".

Further interesting results in this direction were obtained by many mathematicians, and bellow we present some results having a direct relation to the present work [3–5].

Let $\Phi = \{\varphi_k(x)\}$ be the Walsh system and let $f(x,y) \in L^p$, $p \geq 1$. We denote by $c_{k,n}(f)$ the Fourier coefficients of $f(x,y)$ with respect to the Walsh double system, i.e.

$$c_{k,n}(f) = \int_0^1 \int_0^1 f(t,\tau)\varphi_k(t)\varphi_n(\tau)dtd\tau, \text{ where } k,n = 0,1,2,\dots$$

The spectrum of $f(x,y)$ $\big($denoted by $spec(f)\big)$ is the support of $c_{k,n}(f)$, i.e. the index set, where $c_{k,n}(f)$ is non-zero: $spec(f) = \{(k,n), c_{k,n,}(f) \neq 0\}$.

We will say that the sequence $\{b_{k,n}\}(b_{k,n} \geq 0)$ is *monotonically decreasing over all rays,* if $b_{k_1,n_1} \geq b_{k_2,n_2}$, when $k_2 > k_1, n_2 \geq n_1$ and $b_{k_j,n_j} \neq 0,\ j = 1,2$.

In [4] was proved that for any $0 < \varepsilon < 1,\ p \geq 1$ and each function $f \in L^p[0,1]$ one can find a function $g \in L^p[0,1],\ mes\{x \in [0,1] ;\ g \neq f\} < \varepsilon$ such that the

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sequence $\{|c_k(g)|, k \in spec(g)\}$ is monotonically decreasing, where $c_k(g)$ are the Fourier–Walsh coefficients of g, i.e. $c_k(g) = \int_0^1 g(t)\varphi_k(t)dt,\ k = 0.$

In the present work we prove only the following theorem.

Theorem. For any $0 < \varepsilon < 1,\ p \geq 1$ and each function $f(x,y) \in L^p[0,1]^2$ one can find a function $g \in L^p[0,1]^2,\ mes\{\ (x,y) \in [0,1]^2;\ g \neq f\} < \varepsilon$ such that the sequence $\{|c_{k,n}(g)|,\ (k,n) \in spec(g)\}$ is monotonically decreasing over all rays.

2. Basic Lemmas. The Walsh system, an extension of the Rademacher system, may be obtained in the following manner. Let r be the 1-periodic function, defined on $[0,1)$ by $r(x) = \chi_{[0,1/2)} - \chi_{[1/2,1)}$. The Rademacher system, $R = \left\{r_n(x)\right\}_{n=0}^{\infty}$ is defined as follows: $r_n(x) = r(2^n x)$ for all real numbers x and $n = 0,1,\ldots,$ and, in the ordering employed by Paley [6], the n^{th} element of the Walsh system $\{\varphi_n(x)\}$ is given by $\varphi_n(x) = \prod_{k=0}^{\infty} r_k^{n_k}(x)$, where $\sum_{k=0}^{\infty} n_k 2^k$ is the unique binary expansion of n, with each n_k either 0 or 1.

Lemma 1. Let the numbers $\gamma \neq 0,\ \delta \in (0,1),\ \varepsilon \in (0,1), N > 1$ and $\Delta = \Delta_1 \times \Delta_2 \subset [0,1]^2$ be given. Then there exists a set $E \subset \Delta$, $mes(E) > (1-\delta)|\Delta|$ and a double polynomial in the Walsh system of the form

$$Q(x,y) = \sum_{k,n=N}^{M} c_{k,n}\varphi_k(x)\varphi_n(y)$$

satisfying the following conditions:

1. $|c_{k,n}| < \varepsilon$ for all $k,n \in [N,M]$;
2. $Q(x,y) = \gamma\chi_\Delta(x,y)$ for all $(x,y) \in E$;
3. $\|Q\|_p < 4\delta^{-\frac{2}{q}}\|\gamma\chi_\Delta(x,y)\|_p \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right)$;
4. the sequence $\{|c_{k,n}|\}$ is monotonically decreasing over all rays.

By virtue of the Lemma 3 of [4], one can find measurable sets $E_j \subset \Delta_j, j = 1,2,$ and polynomials

$$Q_1(x) = \sum_{k=N}^{M} a_k\varphi_k(x),\ Q_2(y) = \sum_{n=N}^{M} b_n\varphi_n(y),$$

where $|a_k| < \varepsilon,\ \ |b_n| < 1$ for all $k,n \in [N,M]$ and non-zero coefficients in $\left\{|a_k|\right\}_{k=N}^{M}$ and in $\left\{|b_n|\right\}_{n=N}^{M}$ are in decreasing order, $Q_1(x) = \gamma\chi_{\Delta_1}(x), \forall x \in E_1$, $mes(E_1) > \left(1 - \frac{\delta}{2}\right)|\Delta_1|,\ Q_2(y) = \chi_{\Delta_2}(y),\ \forall x \in E_2,\ mes(E_2) > \left(1 - \frac{\delta}{2}\right)|\Delta_2|,$

$$\|Q_1\|_p < 2\delta^{-\frac{1}{q}}\|\gamma\chi_{\Delta_1}(x)\|_p,\ \|Q_2\|_p < 2\delta^{-\frac{1}{q}}\|\chi_{\Delta_2}(y)\|_p.$$

We put

$$E = E_1 \times E_2,\ Q(x,y) = Q_1(x)Q_2(y) = \sum_{k,n=N}^{M} c_{k,n}\varphi_k(x)\varphi_n(y),\ c_{k,n} = a_k b_n.$$

It is easy to notice that E and $Q(x,y)$ will satisfy to the conditions of Lemma 1 . $\square$

Lemma 2. Let the numbers $p \geq 1$, $m_0 > 1$, positive ε and δ and Walsh double polynomial $f(x,y)$ be given. Then one can find a set $E \subset [0,1]^2$, $mes(E) > 1 - \delta$ and a double polynomial in the Walsh double system of the form

$$Q(x,y) = \sum_{k,n=m_0}^{N} c_{k,n}\varphi_k(x)\varphi_n(y),$$

satisfying the following conditions:

1) $|c_{k,n}| < \varepsilon$ for all $k,n \in [m_0,N]$,

2) $Q(x,y) = f(x,y)$ for all $(x,y) \in E$,

3) $\|Q\|_p < 4\delta^{-\frac{2}{q}}\|f\|_p \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right)$,

4) the sequence $\{|c_{k,n}|\}$ is monotonically decreasing over all rays.

Proof. Let

$$f(x,y) = \sum_{k=0}^{M} b_{k,n}\varphi_k(x)\varphi_n(y) = \sum_{\nu=1}^{\nu_0} \gamma_\nu \chi_{\Delta_\nu}(x,y) \ , \ \sum_{\nu=1}^{\nu_0} |\Delta_\nu| = 1 \ ,$$

where $\Delta_\nu = \Delta_\nu^{(1)} \times \Delta_\nu^{(2)}$and $\Delta_\nu^{(j)}$, $\nu = 1,...,\nu_0$, are dyadic intervals .

Applying repeatedly Lemma 1, one can find a sequence of measurable sets $\left\{E_\nu\right\}_{\nu=1}^{\nu_0}$, $E_\nu \subset \Delta_\nu$, $mes(E_\nu) > (1-\delta\ )|\Delta_\nu|$ and polynomials

$$Q_\nu(x,y) = \sum_{k,n=m_{\nu-1}}^{m_\nu-1} c_{k,n}\varphi_k(x)\varphi_n(y) \ , \quad \nu = 1,...,\nu_0 \ ,$$

which satisfy the following conditions:

the sequence $\left\{|c_{k,n}|\right\}_{k,n=m_{\nu-1}}^{m_\nu-1}$ is monotonically decreasing over all rays and $\max_{k,s\in[m_{n-1};m_n)} |c_{k,s}| < \min_{k,s\in(m_n;m_{n+1})} |c_{k,s}| < 2^{-n}\ \varepsilon$

$$Q_\nu = \begin{cases} \gamma_\nu : & \text{if } x \in E_\nu, \\ 0 : & \text{if } x \notin \Delta_\nu, \end{cases}$$

$$||Q_\nu||_p = \left(\int_0^1 \int_0^1 |\ Q_\nu(x,y)\ |^p\ dxdy\right)^{1/p} < 4\delta^{-\frac{2}{q}}|\gamma_\nu\ |\ |\ \Delta_\nu\ |^{1/p} \ .$$

We define

$$E = \bigcup_{\nu=1}^{\nu_0} E_\nu, Q(x,y) = \sum_{\nu=1}^{\nu_0} Q_\nu(x,y) = \sum_{k,n=m_0}^{N} c_{k,n}\varphi_k(x)\varphi_n(y) \ , \ N = m_{\nu_0} - 1 \ .$$

It is not hard to notice that the sequence $\{|c_{k,n}|\}$ monotonically decreases over all rays and

$$Q(x,y) = f(x,y), \quad \text{for } (x,y) \in E,\ mes(E) > 1-\delta\ \ , \|Q\|_p < 4\delta^{-\frac{2}{q}}\|f\|_p. \quad \square$$

3. Proof of Theorem.

Let $p \geq 1,\ f(x,y)$ be an arbitrary element of $L^p[0,1]^2$, and let $\varepsilon \in (0,1)$. It is not hard to choose a sequence $\{f_n(x,y)\}_{n=1}^{\infty}$ of polynomials in the Walsh double systems such that

$$\lim_{N\to\infty}\left|\left|\sum_{n=1}^{N} f_n(x,y)-f(x,y)\right|\right|_p = 0\ , ||f_n(x,y)||_p \leq \varepsilon^{\frac{2}{q}}2^{-4(n+1)},$$
$$n \geq 2\ \left(\frac{1}{p}+\frac{1}{q}=1\right).$$

Applying repeatedly Lemma 2, we obtain a sequence of sets $\{E_n\}_{n=1}^{\infty}$ and polynomials in the Walsh double systems $\{\varphi_k(x)\varphi_s(y)\}$
$Q_n(x,y) = \sum\limits_{k,s=m_{n-1}}^{m_n-1} a_{k,s}\varphi_k(x)\varphi_s(y)\ ,\ n \geq 1,\ m_n \nearrow$, which for all $n \geq 1$ satisfy the following conditions:

$$Q_n(x,y) = f_n(x,y)\ ,\ \text{for } (x,y) \in E_n, |E_n| > 1-\varepsilon 2^{-n},\ ||Q_n||_p \leq 4\varepsilon^{-\frac{2}{q}}2^{\frac{2}{q}}||f_n||_p\ ,$$

the sequences $\left\{|a_{k,s}|\right\}_{k,s=m_n}^{m_{n+1}}$ and $\left\{|a_{k,s}|\right\}_{k,s=m_{n-1}}^{m_n}$ are monotonically decreasing over all rays and $\max_{k,s\in[m_{n-1};m_n)}|a_{k,s}| < \min_{k,s\in[m_n;m_{n+1})}|a_{k,s}| < 2^{-n}$.

We put

$$g(x,y) = \sum_{n=1}^{\infty} Q_n(x,y) = \sum_{k,s=0}^{\infty} a_{k,s}\varphi_k(x)\varphi_s(y)\ .$$

Obviously, $g(x,y) \in L^p[0,1]^2$,

$$g(x,y) = f(x,y),\ \text{ for } (x,y) \in \bigcap_{n=1}^{\infty} E_n,\ \ mes(\bigcap_{n=1}^{\infty} E_n) > 1-\varepsilon.$$

It is not hard to see that $a_{k,n} = c_{k,n}(g) = \int_0^1\int_0^1 g(t,\tau)\varphi_k(t)\varphi_n(\tau)dtd\tau$, $k,n = 0,1,2,...$ and $\{|c_{k,n}(g)|,\ (k,n) \in spec(g)\}$ decreases over all rays.

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