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Sharp Theorems on Traces in Analytic Spaces in Bounded Strictly Pseudoconvex Domains

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1.Introduction and preliminaries. Let Ω be bounded strongly pseudoconvex domain in $\mathbb{C}^n$. By δ we denote a positive function - the Euclidean distance from the boundary of Ω, that is $\delta(z) = d(z,\Omega_1)$, where Ω_1 is the boundary of domain Ω. Let further $H(\Omega)$ denote the space of all holomorphic functions on Ω endowed as usual with the topology of uniform convergence on compact subsets. $V(z)$ will be the Lebesgue measure on domain Ω. $K(z,w)$ or $K_{n+1}(z,w)$ will be the Bergman kernel of order $n+1$ of Ω (see [2], [4]). For positive α the Bergman kernel of order α will be denoted by $K_\alpha(z,w)$ (see [2], [4]). For given $r \in (0,1)$ and $z_0 \in \Omega$, we shall denote by $B_\Omega(z_0,r)$ the Kobayashi ball of center z_0 and the radius $\frac{1}{2}\log\frac{1+r}{1-r}$. We refer the reader [2] for definitions, basic properties and applications to geometric function theory of the Kobayashi distance and for definition and basic properties of Bergman kernel. We will need several results for our proofs about Kobayashi balls for which we refer the reader to [1], [2]. In particular we constantly use among other things the fact that $V(B_\Omega(z,r))$ is equivalent to $\delta^{n+1}(z)$, $r \in (0,1)$.

We denote n products of such domains by Ω^n, the space of all analytic function on this new product domain which are analytic by each variable

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separately will be denoted by $\mathrm{H}(\Omega^n)$. In this paper we will be interested on properties of certain analytic subspaces of $\mathrm{H}(\Omega^n)$. Note it is known from general theory that the product of final number of strongly pseudoconvex domains is a pseudoconvex domain and hence obviously only specific problems in such case can be interesting and can be considered and solved in this setting. Probably for the first time products of strongly (or strictly) pseudoconvex domains and various properties of analytic functions on them were considered in [11]. On the other hand various other type extension theorems in such domains in $\mathbb{C}^n$ were studied and proved by many authors (see [3], [4], [5], [7], and various references there). Our results will add a new page to this list of known results. Let

$$A_\tau^\infty(\Omega) = \left\{ F \in \mathrm{H}(\Omega) : \|F\|_{A_\tau^\infty} = \sup_{z\in\Omega} |F(z)|\,\delta^\tau(z) < \infty \right\}, \qquad (1)$$

(see [2], [9] and references there). It can be checked that this is a Banach space.

For $1 < p < +\infty$ and $\nu \in R$ and $\nu_j > -1$ we denote by $A_\nu^p(\Omega)$ the weighted Bergman space consisting of analytic functions f in Ω such that

$$\|F\|_{A_\nu^p}^p = \left(\int_\Omega |F(z)|^p \, dV_\nu(z) \right) < \infty.$$

Here we used the notation $dV_\nu(w) = \delta^\nu dV(w)$. Below we use also the following notations $w = u + iv \in \Omega$ and also $z = x + iy \in \Omega$. This space is a Banach space. Replacing above simply A by L we will get, as usual, the corresponding larger spaces of all measurable functions in the same domain with the same norm (see [1], [2]).

To define related two Bergman-type spaces $A_\nu^p(\Omega)$ and $A_\tau^\infty(\Omega^n)$ (ν and τ can be also vectors) in m-products of tube domains Ω^n we follow standard procedure which is well-known in the case of unit disk and unit ball (see [10], [12], [13], [14], [16]). Namely we consider analytic separately by each variable F functions $F = F(z_1, \dots z_m)$, where each variable belongs to tube Ω. It can be shown that these are also Banach spaces. Replacing above simply A by L we will get as usual the corresponding larger space of all measurable functions in products of bounded strictly pseudoconvex domains with smooth boundary with the same norm. The (weighted) Bergman projection P_ν is the orthogonal projection from the Hilbert space $L_\nu^2(\Omega)$ onto its closed subspace $A_\nu^2(\Omega)$ and it is given by the following integral formula (see, for example, [2], [4], [9]).

$$P_\nu f(z) = C_\nu \int_\Omega K_{\nu+n+1}(z,w) f(w) dV_\nu(w), \qquad (2)$$

where $K_{\nu+n+1}(z,w)$ is the Bergman reproducing kernel for $A_\nu^2(\Omega)$ (see [1], [2]), where $dV_\nu(w) = \delta^\nu dV(w)$, $\nu > -1$.

For any analytic function from $A_\nu^2(\Omega)$ the following integral formula is valid (see also [2], [4])

$$f(z)=C_\nu\int_\Omega K_{\nu+n+1}(z,w)f(w)dV_\nu(w). \quad (3)$$

The existence of suitable covering of domain D (r-lattice) based on Kobayashi balls is crucial for results of this note. Let D be a bounded domain and $r>0.$ An r-lattice in D is a sequence $(a_k),$ $(a_k)\in D,$ so that each point $z\in D$ belongs to at least one ball $B_D(a_k,r),$ and there exists $m>0$ such that any point in D belongs to at most m balls of the form $B_D(a_k,r),$ where $R=\frac{1}{2}(1+r).$ There is such a r-lattice in each bounded strictly pseudoconvex domain with smooth boundary [2]. In this case sometimes below we say sometimes below simply that the f function allows the Bergman representation via Bergman kernel with ν index. Note that these assertions have direct copies in simpler cases of analytic function spaces in unit disk, polydisk, unit ball, upperhalfspace C_+ and in spaces of harmonic functions in the unit ball or upperhalfspace of the Euclidean space $R^n.$ These classical facts are well-known and can be found, for example, in [6], [9], [16] and in some items from references there.

We will need for our proofs the following important fact on integral representations (see[4], [9]). For all $1\le p<\infty,$ $\nu>-1$ and for all f functions from A_ν^p the Bergman representation formula with $\alpha+n+1$ index or with the Bergman kernel $K_{\alpha+n+1}(z,w)$ is valid, for all $\alpha,$ $\alpha>\alpha_0,$ for certain fixed $\alpha_0,$ [2], [4], [9]. Let $\alpha>-1$ then for all $\nu>\nu_0$ for certain fixed ν_0 and all f functions $f\in A_\alpha^\infty$ the integral representations of Bergman with Bergman kernel $K_{\nu+n+1}(z,w)$ (with $\nu+n+1$ index) is valid. We note also that (see [2], [4], [9])

$$|f(z)|\delta^{\frac{n+1+\nu}{p}}(z)\le c_{p,\nu}\|f\|_{A_\nu^p}, \quad z\in\Omega. \quad (4)$$

All the mentioned results together with properties of the Whitney decomposition (r-lattice) of bounded strictly pseudoconvex domains with smooth boundary based on Kobayashi balls in $C^n.$ [1], [2], [9] are used heavily during all proofs of our assertions.

2. On sharp estimates for traces in analytic function spaces in bounded strictly pseudoconvex domains in $\mathbb{C}^n$. Theorem 1. *Let* $f\in A_{\vec\nu}^p(\Omega^n),$ $1\le p<\infty,$ $\nu_j>-1,$ $j=1,\dots,m.$ *Then* $f(z,\dots,z)\in A_s^p,$ *where* $s=(\sum_{j=1}^m\nu_j)+(n+1)(m-1).$ *And the reverse is also true. For each* g *function* $g\in A_s^p(\Omega)$ *there is an* F *function* $F(z,\dots,z)=g(z),$ $F\in A_{\vec\nu}^p(\Omega^n).$ *Let in addition*

$$T_\beta f(z_1,\dots,z_m)=C_\beta\int_\Omega f(w)\prod_{j=1}^m\left|K_{\beta+n+1}^t(z_j,w)\right|dV_\beta(w),$$

$mt=1,\ \ z_j\in\Omega,\ \ j=1,\dots,m.$ *Then the following assertion holds for all* β, $\beta>\beta_0$ *for certain positive* β_0. *The* T_β *Bergman type integral operator (expanded Bergman projection) maps* $A_s^p(\Omega)$ *to* $A_\nu^p(\Omega^n)$, $\nu=(\nu_1,\dots,\nu_m)$. *Let* $1\le p<\infty$, $F\in H(\Omega^n)$. *Then* $F(z,\dots,z)\in A_s^p$ *if* $F\in A_\nu^p$ *with related estimate for norms.*

We remark that in particular case of unit ball in $\mathbb{C}^n$ this theorem can be found in [12], [15] and in the case of the unit disk in various papers (see, for example, [8], [16] and various references there). A complete analogue of this theorem is true also for the case $p=\infty$.

Theorem 2. *Let* $f\in A_\nu^\infty(\Omega^n)$, $\nu\in R^n$, $\nu_j>0$, *for all* $j=1,\dots,m$. *Then* $f(z,\dots,z)\in A_s^\infty$, *where* $s=\sum_{j=1}^m\nu_j$. *And the reverse is also true. For each* g *function* $g\in A_s^\infty(\Omega)$ *there is an* F *function,* $F(z,\dots,z)=g(z)$, $F\in A_\nu^\infty(\Omega^n)$. *Let in addition*

$$T_\beta f(z_1,\dots,z_m)=C_\beta\int_\Omega f(w)\prod_{j=1}^m\left|K_{\beta+n+1}^t(z_j,w)\right|dV_\beta(w),$$

$mt=1,\ \ z_j\in\Omega,\ \ j=1,\dots,m.$ *Then the following assertion holds for all* β, $\beta>\beta_0$, *for certain positive* β_0. *The* T_β *Bergman-type integral operator (expanded Bergman projection) maps* $A_s^\infty(\Omega)$ *to* $A_\nu^\infty(\Omega^n)$, $\nu=(\nu_1,\dots,\nu_m)$.

To define the next space based on Kobayashi balls we remind the reader that there exists a family of Kobayashi balls $B_\Omega(a_k,r)$ which forms an r-lattice in bounded strictly pseudoconvex domain Ω (see [1], [2], [4]). We denote by $B_\Omega^m(z,r)$ m-products of such Kobayashi $B_\Omega(z,r)$ balls in $\mathbb{C}^m$, where $z\in\Omega$. . By $M_{\nu,\tau}^p(\Omega^n)$ we denote all f analytic functions in Ω^n so that

$$\int_{B_\Omega^m(Z,r)}|f(z_1,\dots,z_m)|^p\prod_{j=1}^m\delta^{s_j}(z_j)dz_j$$

belongs to $L^1_{\tau_1,\dots,\tau_m}(\Omega^n)$, where $s_j>-1$ for all $j=1,\dots,m,$ $1\le p<\infty$, $\tau_j>-1$, for all $j=1,\dots,m.$ Note in polyball complete analogues of these classes were considered in [13] and the complete description of Traces of these spaces were also given. We obtain below a complete extension of that result to the case of bounded strictly pseudonconvex domains with smooth boundary.

Theorem 3. *Let* $1\le p<\infty$, $f\in M_{\nu,\tau}^p(\Omega^n)$. *Then* $f(z,\dots,z)$ *belongs to* $A_s^p(\Omega)$, $1\le p<\infty$, $s=\sum_{j=1}^m(\nu_j+\tau_j)+(n+1)(2m-1)$. *For every* f *function* $f\in A_s^p$ *there is an* F *function* $F\in M_{\nu,\tau}^p$, *so that* $F(z,\dots,z)=f(z)$. *Let in addition*

$$T_\beta f(z_1,\ldots,z_m) = C_\beta \int_\Omega f(w) \prod_{j=1}^{m} \left| K^t_{\beta+n+1}(z_j, w) \right| dV_\beta(w),$$

$mt = 1, \quad z_j \in \Omega, \quad j = 1,\ldots,m.$ *Then the following assertion holds for all* β, $\beta > \beta_0$ *for certain positive* β_0. *The* T_β *Bergman-type integral operator (expanded Bergman projection) maps* $A^p_s(\Omega)$ *to* $M^p_{\nu,\tau}(\Omega^n)$, $\nu = (\nu_1,\ldots,\nu_m)$, $\tau = (\tau_1,\ldots,\tau_n)$.

All proofs are based on vital estimates which were obtained recently in [1], [2]. Complete analogues of all our assertions in disk, polyball can be found in [12], [13], [14], [16].

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Sharp Theorems on Traces in Analytic Spaces in Bounded Strictly Pseudoconvex Domains

New sharp estimates of traces in Bergman-type spaces of analytic functions in bounded strictly pseudoconvex domains are obtained. These are, as far as we know, for the first time results of this type which are valid for any bounded strictly pseudoconvex domains.

Р. Ф. Шамоян, О. Р. Милич

Точные теоремы о следах аналитических пространств в строго псевдовыпуклых областях

Получены новые точные оценки для следов пространств типа Бергмана аналитических функций в ограниченных строго псевдовыпуклых областях. Насколько нам известно, это первые результаты такого типа для псевдовыпуклых областей с гладкими границами.

Ռ. Ֆ. Շամոյան, Օ. Ռ. Միլիչ

Ճշգրիտ թեորեմներ խիստ պսևդոուռուցիկ տիրույթների մեջ անալիտիկ տարածությունների հետքերի մասին

Ստացված են նոր ճշգրիտ գնահատականներ Բերգմանի տիպի անալիտիկ ֆունկցիաների տարածությունների համար խիստ պսևդոուռուցիկ սահմանափակ տիրույթների համար: Որքան մեզ հայտնի է, դրանք այդ տեսակի առաջին

արդյունքներն են ողորկ եզրերով պսևդոուռուցիկ սահմանափակ տիրույթների համար:

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