PROCEEDINGS OF NPUA. MECHANICS, MACHINE SCIENCE, MACHINE-BUILDING, 2016.№2

MACHINE SCIENCE

UDC 620.178.153

RIGID BODY POINTS APPROXIMATING COAXIAL CYLINDERS IN ALTERNATING SETS OF ITS POSITIONS

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The problem of determining special points of a moving body which in alternating sets of its given positions deviate least, in the least square sense, from coaxial circular cylinders is considered. The sought-for approximation is one which minimizes the sum of squared algebraic deviations of these points from the coaxial cylinders approximating their paths in each of the given sets of positions. The points of interest lie at the intersection of three 11^{th} order surfaces determined from the stationary conditions of the least square objective function. The theory and methods developed in this paper can be readily applied to the synthesis of adjustable parallel robotic mechanisms of modular structure designed for the approximate generation of multiphase motions or multiple point-paths of the output link.

Keywords: coaxial cylinders, least square approximation, algebraic deviation, approximate synthesis, reconfigurable robotic mechanism.

Introduction. In [1] the points of a rigid body whose several positions lie on a right-circular cylinder with the specified direction of its axis have been studied. It was established that under a general motion there are no points with more than 6 positions on one cylinder with the given axis orientation. In the general case, when the axis direction is left unspecified, the maximum number of such positions can reach 8. In [2] we have first considered the problem of determining those points in a moving body which, in an unlimited number of its prescribed positions, remain as close as possible to a cylinder, using as the measure of closeness the mean square deviation of such points from a cylinder for all given rigid body positions. The results of this study are presented also in [3], as a part of the approximational kinematic geometry developed there.

This paper is an extension of [2] for multiple sets of the prescribed positions of a moving body. Here we seek to determine those points of the moving body which in given sets of positions approximate, in the least square sense, a family of coaxial cylinders corresponding to different sets of the given positions. Apart from its theoretical interest, the results of the present paper have also direct applications in designing adjustable robotic mechanisms built of SPC type dyadic modules, connecting the manipulation object with the frame. A general methodology of the approximational synthesis of adjustable robotic mechanisms with different structural alternatives of their dyad-modules is described in [4].

The paper is organized by the following 2 phases. First we consider a simpler problem of determining least square approximations of the given position sets of a fixed point in the moving body by coaxial cylinders with the given direction of their common axis. Then we use the closed form solution of this problem as a basis for the search of the points in the moving body which fit best to coaxial cylinders in the multiple sets of its given positions.

Least square approximations of alternating point-position sets by coaxial circular cylinders. A rigid body *e* undergoes spatial motion with respect to a fixed body *E* given by *m* sets of finitely separated positions $\{e_{ji}\}_{i=1}^{N_j}(j=1,2,...,m)$. Coordinate systems oxyz and OXYZ are rigidly attached to *e* and *E* respectively. We consider the following problem: given a point $B(x_B, y_B, z_B)$ in *e*, determine a set of *m* circular cylinders $\{C_{ji}\}_{j=1}^{m}$ with the common axis $Q \in E,$ so that each *j*-th cylinder C_j of this set is as close as possible to the given *j*-th point-set $\{B_{ji}\}_{i=1}^{N_j}$.

For each cylinder C_j there is a point-set $\{B_{ji}\}_{i=1}^{N_j}$ which should be "close" to that cylinder. In other words, we need to find one axis *Q* and *m* radii h_j around it that will fit all *m* given point-sets as closely as possible. This problem would have been separable to *m* independent cylinder fitting problems considered in [2]. However, we require a common axis *Q* for all cylinders.

A brief review of the known curve and surface fitting methods shows that the objective functions estimating "closeness" of the given point-sets to a cylinder usually have 3 formulations:

1) The least square objective when the sum of squares of the distances of points B_{ji} to the cylinders C_j are minimized.
2) The minimax objective when the maximum distance of points B_{ji} to the cylinders C_j is minimized.
3) The minisum objective when the sum of these distances is minimized.

By analogy with [2], here we will use the least square objective, leading to a closed form solution of the problem under consideration.

The distance of point B_{ji} from the cylinder axis *Q* can be represented in the following form:

$$h_{ji} = \frac{\left|(\overline{R_{B_{ji}}} - \overline{R_A}) \times \overline{Q}\right|}{Q}, \tag{1}$$

where $\overline{Q}(Q_x, Q_y, Q_z)$ is the orientation vector of the axis *Q*, $\overline{R_{B_{ji}}}$ is the position vector of B_{ji} from the origin O, $\overline{R_A}$ is the position vector of the intersection point *A* of *Q* with coordinate plane OXZ. With these denotations the normal distance δ_{ji} of B_{ji} from *Q* may be written as:

$$\delta_{ji}=h_{ji}-h. \qquad (2)$$

However, there is no closed form solution for the least square cylinder fitting problem based on (2), since the unknown axis coordinates are under radicals and lengthy iterative search routines are needed to determine them.

The denominator in (1) does not depend on the position of e in E. It permits us to use another expression known as algebraic deviation or weighed difference as the error function [3]:

$$\Delta_{ji}=Q^2\left(h_{ji}^2-h^2\right)=\left(\overline{R_{B_{ji}}}\times\overline{Q}\right)^2+2\left[\left(\overline{R_{B_{ji}}}-\overline{R_A}\right)\times\overline{Q}\right]\overline{R_A}+\left(\overline{R_A}\times\overline{Q}\right)^2-Q^2h_j^2. \qquad (3)$$

Transforming (3) and grouping for X_A, Z_A and constant H, we obtain the linear function

$$\Delta_{ji}=f_{ji}^{(1)}X_A+f_{ji}^{(2)}Z_A+H_j+f_{ji}^{(3)}, \qquad (4)$$

where

$$f_{ji}^{(1)}=2\left(Q_z\cdot U_{jiY}-Q_YU_{jiZ}\right),\qquad f_{ji}^{(2)}=2\left(Q_Y\cdot U_{jiX}-Q_XU_{jiY}\right),$$

$$f_{ji}^{(3)}=U_{jiX}^2+U_{jiY}^2+U_{jiZ}^2,\qquad H_j=\left(\overline{R_A}\times\overline{Q}\right)^2-Q^2h_j^2$$

and U_{jiX}, U_{jiY}, U_{jiZ} are the projections of vector $\overline{U}_j=\overline{R}_{B_{ji}}\times\overline{Q}$ on the fixed coordinate axes:

$$U_{jiX}=\begin{vmatrix}Y_{B_{ji}} & Z_{B_{ji}}\\ Q_y & Q_Z\end{vmatrix},\qquad U_{jiY}=\begin{vmatrix}Z_{B_{ji}} & X_{B_{ji}}\\ Q_z & Q_x\end{vmatrix},\qquad U_{jiZ}=\begin{vmatrix}X_{B_{ji}} & Y_{B_{ji}}\\ Q_x & Q_y\end{vmatrix}.$$

The selected least square objective requires to minimize the sum of the squared errors (4) for all $N=\sum_{j=1}^{N_j}N_j$ given positions of e:

$$S=\sum_{j=1}^{m}\sum_{i=1}^{N_j}\Delta_{ji}^2. \qquad (5)$$

For any point $B\left(x_B,y_B,z_B\right)$ in e and specified direction of Q, the coaxial cylinders $C_j\left(j=1,2,\ldots,m\right)$ approximating the corresponding sets of point positions $\left\{B_{ji}\right\}_{i=1}^{N_j}\left(j=1,2,\ldots,m\right)$ are determined from the necessary conditions for a minimum of (5):

$$\frac{\partial S}{\partial X_A}=0,\ \frac{\partial S}{\partial Z}=0,\ \frac{\partial S}{\partial h_1}=0,\ \ldots,\ \frac{\partial S}{\partial h_m}=0. \qquad (6)$$

By substituting in (5) from (4), after some transformations, conditions (6) can be reduced to the following system of ($2+m$) linear equations in X_A, Z_A, H_j $\left(j=1,2,\ldots,m\right)$ presented below in a matrix form:

$$M\cdot P=F, \qquad (7)$$

where

$$M=\begin{bmatrix} \sum_{j=1}^{m}\sum_{i=1}^{N_j}\left(f_{ji}^{(1)}\right)^2 & \sum_{j=1}^{m}\sum_{i=1}^{N_j} f_{ji}^{(1)} f_{ji}^{(2)} & \sum_{i=1}^{N_1} f_{1i}^{(1)} & \dots & \sum_{i=1}^{N_m} f_{mi}^{(1)} \\ \sum_{j=1}^{m}\sum_{i=1}^{N_j} f_{ji}^{(1)} f_{ji}^{(2)} & \sum_{j=1}^{m}\sum_{i=1}^{N_j}\left(f_{ji}^{(2)}\right)^2 & \sum_{i=1}^{N_1} f_{1i}^{(2)} & \dots & \sum_{i=1}^{N_m} f_{mi}^{(2)} \\ \sum_{i=1}^{N_1} f_{1i}^{(1)} & \sum_{i=1}^{N_1} f_{1i}^{(2)} & N_1 & \dots & 0 \\ \sum_{i=1}^{N_2} f_{2i}^{(1)} & \sum_{i=1}^{N_2} f_{2i}^{(2)} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^{N_m} f_{mi}^{(1)} & \sum_{i=1}^{N_m} f_{mi}^{(2)} & 0 & \dots & N_m \end{bmatrix},$$

P = [X_A, Z_A, H_1, ... ,H_m],

$$F=\left[\sum_{j=1}^{m}\sum_{i=1}^{N_j} f_{ji}^{(1)} R_{B_{ji}}^2, \sum_{j=1}^{m}\sum_{i=1}^{N_j} f_{ji}^{(2)} R_{B_{ji}}^2, \sum_{i=1}^{N_1} R_{B_{1i}}^2, \dots, \sum_{i=1}^{N_m} R_{B_{mi}}^2\right].$$

It is easy to see from (3–5) that conditions $\partial S/\partial h=0$ in (6) are equivalent to $\partial S/\partial H_j=0$ and thus lead to the same last *m* linear equations in (7).

To solve system (7), we express coordinates $X_{B\,ji}, Y_{B\,ji}, Z_{B\,ji}$ of point *B* in *E* through its coordinates in *e* by means of the following linear transformation:

$$\begin{bmatrix} X_{B\,ji} \\ Y_{B\,ji} \\ Z_{B\,ji} \end{bmatrix}=\begin{bmatrix} X_{O\,ji} \\ Y_{O\,ji} \\ Z_{O\,ji} \end{bmatrix}+T_{ji}\begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}, \tag{8}$$

where T_{ji} is a 3×3 rotation matrix which brings the system xyz from a position with its axes initially parallel to the axes *X*, *Y*, *Z* to its *i*-th position of the given *j*-th position set $\{e_{ji}\}_{i=1}^{N_j}$.

It follows then that if we have *m* sets of finite positions e_{ji}(j=1,2,...,m; i=1,2,…,N_j) and specify a point $B(x_B, y_B, z_B)$ in *e*, equation (7) uniquely determines *m* right circular cylinders $C_j\,(j=1,2,...,m)$ with their common axis *Q* parallel to the given vector $\overline{Q}(Q_X, Q_Y, Q_Z)$ and radii h_j ($j=1,2,…,m$) given by the expression

$$h_j = \left[(\overline{R}_A \times \overline{Q}) - H_j\right]^{\frac{1}{2}} \; (j=1,2,...,m). \tag{9}$$

For the further analysis, it is convenient to present the solution of (7) in the vector form:

$$(X_A, Z_A, H_1,..., H_m) = \frac{1}{D}(D_A, D_A, D_{H_1},..., D_{H_m}), \tag{10}$$

where *D* is the coefficient determinant of (7), $D_A, D_A, D_{H_1},..., D_{H_m}$ are *(2+m)*-th order minors in the expanded matrix of (7).

It was established in [2] that if the total number of the given positions N>5, D>0 for each point $B \in e$. Determinants in the right side of (10) are functions of coordinates x_B, y_B, z_B of *B* in *e*. It follows then from (10) that corresponding to any point B in e there is a unique line *Q* in *E* which is the common axis of cylinders C_j *(j=1,2,…,m)* (with the varying radii (9)) approximating, in the least square sense, given m alternating point-sets $\{B_{ji}\}_{i=1}^{N_j}(j=1,2,...,m)$. Now, if we fix a line *Q* in *E* with the given orientation (Q_x, Q_y, Q_z), double sum (5) can be transformed with the use of (8) into a *4*-th order polynomial in x_B, y_B, z_B. Thus, the stationary conditions for S: *4*-th order polynomial in x_B, y_B, z_B. Thus, the stationary conditions for S: $\partial S/\partial x_B = 0,\ \partial S/\partial y_B = 0, \partial S/\partial z_B = 0$ generate 3 cubic surfaces in *e*. It follows then that for any axis *Q* in *E* with the fixed orientation (Q_x, Q_y, Q_z) we can find at most 27 real points in *e* which make objective function (5) stationary.

Points of e deviating least from coaxial cylinders in the given position sets $\{\mathbf{e_{ji}}\}_{\mathbf{i=1}}^{\mathbf{N_j}}(j=1,2,...,m)$**.** Now we proceed to the main issue of this study: which points of *e* will approximate best coaxial cylinders C_j *(j=1,2,…,m)* in the corresponding position-sets $\{e_{ji}\}_{i=1}^{N_j}(j=1,2,...,m)$? Given the position e_{ji} of *e* and the fixed orientation of *Q*, double sum (5) becomes a function of *(5+m)* design variables: $X_A, Z_A, x_B, y_B, z_B, H_j\ (j=1,2,...m)$.

Therefore, for *S* to be a minimum, the following conditions are necessary:

$$\partial S/\partial X_A = 0,\ \ \partial S/\partial Z_A = 0, \partial S/\partial x_B,\ \partial S/\partial y_B,\ \partial S/\partial z_B,\ \ \partial S/\partial H_j = 0\ (j=1,2,...,m). \tag{11}$$

Equation (10) following from the system of the first two and last *m* conditions (11) together with (8) permits to express parameters $X_A, Z_A, H_j\,(j=1,2,...,m)$ through the sought-for coordinates x_B, y_B, z_B. To study the locus of points in *e* for which *S* has stationary values the 3-th, 4-th and 5-th conditions in (11) are presented as:

$$\sum_{j=1}^{m}\sum_{i=1}^{N_j}\Delta_{ji}\frac{\partial\Delta_{ji}}{\partial x_B}=0,\ \sum_{j=1}^{m}\sum_{i=1}^{N_j}\Delta_{ji}\frac{\partial\Delta_{ji}}{\partial y_B}=0,\ \sum_{j=1}^{m}\sum_{i=1}^{N_j}\Delta_{ji}\frac{\partial\Delta_{ji}}{\partial z_B}=0. \qquad (12)$$

Using expressions (4), (8), (10), after some transformations, equations (12) can be presented in the following form [2]:

$$\begin{aligned} &k_1^l D_X^2 + k_2^l D_Z^2 + k_3^l D_X D_Z + \sum_{j=1}^{m} k_{4j}^l D_X D_Z + \sum_{j=1}^{m} k_{5j}^l D_X D_Z + k_6^l D_X D + \\ &+ k_7^l D_Z D + \sum_{j=1}^{m} k_{8j}^l D_{H_j} D + k_9^l D^2 = 0 \quad (l = x, y, z), \end{aligned} \qquad (13)$$

where coefficients $k_1^l, ..., k_9^l \quad (l = x, y, z)$ are zero, first, second or third order polynomials in x_B, y_B, z_B.

Equations (13) define three 11-th order surfaces in *e* whose common points satisfy all *(5+m)* conditions (11) necessary for a minimum of the objective function (5). As established in [2] for the case *m=1*, the maximum number of real common points of these surfaces is 211. It is easy to make sure that this result does not depend on *m* and is true also for any *m>1*. The real task is to distinguish among so many possible solutions of (13) those which provide a sufficient accuracy of approximation to coaxial circular cylinders.

For practical aims, it may be more effective to avoid the cumbersome computational process of solving nonlinear equations (13) and organize an iterative search of the sought-for points in *xyz* system based on the direct minimization of the mean square sum (5) by one of the computational algorithms developed in [3].

Application to the synthesis of reconfigurable robotic mechanisms. The special points of *e* with the coaxial approximate cylindrical paths traced in alternating sets of its prescribed displacements can be directly applied to design reconfigurable parallel robotic mechanisms for the approximate generation of the given multiple motions of the output link to fit the requirements of the changing kinematic tasks. The procedure of the synthesis of such mechanisms is very simple [4]: in each of the determined special points of *e* which in the *m* given sets of its positions remain sufficiently close to a family of coaxial circular cylinders we attach a two link SPC type kinematic chain (dyad) connected with the frame *E* with its cylindrical pair. The middle prismatic joint of this dyad is fixed in each of the *m* working cycles of the mechanism and functions only in the reconfiguration mode, when the distance from the center of the spherical joint (*S*) to the axis of the cylindrical joint (*C*) is readjusted for the changing kinematic tasks. Attaching to *e* five SPC dyads, we obtaina reconfigurable 5 (SPC) robotic mechanism of modular structure (Fig.) realizing approximately the given *m* sets of positions of the output link *e*.

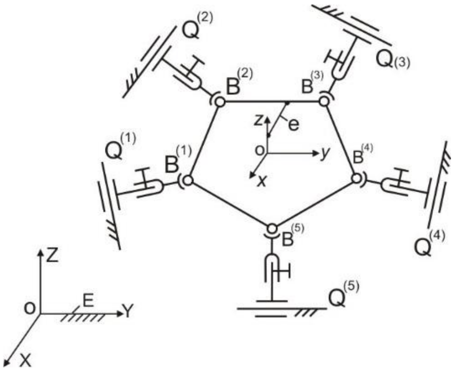

Fig. A reconfigurable parallel robotic mechanism with 5 SPC type dyad-modules

Conclusion. We have presented a study of special points of a rigid body which in the *m* alternating sets of the given positions deviate least from the coaxial circular cylinders. The results presented here can be considered as a generalization of the theory of so called "least square cylindrical points" developed in [2] for the case of a single set of rigid body positions. Similar to the case of a single position set, it is established that the locus of the sought-for special points lies at the intersection of three 11^{th} order algebraic surfaces embedded in *e*, while the maximum number of the real common points of these surfaces cannot be more than 211. The special points studied above can be readily applied for building reconfigurable platform-type manipulators composed of SPC dyads connecting output link *e* with frame *E*. These mechanisms are designed to generate prescribed multiphase motions or multiple paths of the moving platform with the required accuracy of approximation. The numerical results of this study will be presented in a companion paper.

Acknowledgment. This work is supported by the grant 15T-2D044 of SCS, Armenia.

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Received on 17.10.2016.
Accepted for publication on 05.12.2016.

ՊԻՆԴ ՄԱՐՄՆԻ ՏՐՎԱԾ ԴԻՐՔԵՐԻ ԻՐԱՐԱՀԱՋՈՐԴ ԲԱԶՄՈՒԹՅՈՒՆՆԵՐՈՒՄ ՀԱՄԱՌԱՆՑՔ ԳԼԱՆՆԵՐ ՄՈՏԱՐԿՈՂ ԿԵՏԵՐԻ ՄԱՍԻՆ

Յու.Լ. Սարգսյան

Դիտարկվում է տեղաշարժվող պինդ մարմնի այնպիսի կետերի որոշման խնդիրը, որոնք նրա նախանշված դիրքերի իրարահաջորդ բազմություններում ամենաքիչն են շեղվում համառանցք շրջանագծային գլաններից՝ նվազագույն քառակուսիների իմաստով: Հաշվարկվող մոտարկումը նվազարկում է որոնելի կետերի համառանցք գլաններից հանրահաշվական շեղումների քառակուսիների գումարը մարմնի տված դիրքերի բոլոր բազմությունների համար: Ուսումնասիրության առարկա բնութագրական կետերը որոշվում են միջին քառակուսային շեղման նպատակային ֆունկցիայի ստացիոնարության պայմաններն արտապատկերող 11-րդ կարգի երեք հանրահաշվական հավասարումների փոխհատումով: Ներկայացված տեսությունը և մեթոդները կարող են ուղղակիորեն կիրառվել մոդուլային կառուցվածքով կարգավորվող զուգահեռ մանիպուլյացիոն մեխանիզմների սինթեզում, որոնք նախատեսված են ելքի օղակի բազմափուլ շարժումների կամ բազմակի հետագծերի մոտավոր վերարտադրության համար:

Առանցքային բառեր. համառանցք գլաններ, քառակուսային մոտարկում, հանրահաշվական շեղում, ապրոքսիմացիոն սինթեզ, վերակոնֆիգուրացվող մանիպուլյացիոն մեխանիզմ:

ТОЧКИ ТВЕРДОГО ТЕЛА, АППРОКСИМИРУЮЩИЕ СООСНЫЕ ЦИЛИНДРЫ В ЧЕРЕДУЮЩИХСЯ МНОЖЕСТВАХ ЕГО ЗАДАННЫХ ПОЛОЖЕНИЙ

Ю.Л. Саркисян

Рассматривается задача определения точек движущегося тела, которые в чередующихся множествах его положений наименее отклоняются от соосных прямых цилиндров в смысле наименьших квадратов. Искомое приближение минимизирует сумму квадратов алгебраических отклонений (расстояний) указанных точек от соосных цилиндров в соответствующих множествах заданных положений. Интересующие нас характерные точки лежат на пересечении трех алгебраических поверхностей одиннадцатого порядка, отображающих условия стационарности целевой функции среднеквадратического отклонения. Теория и методы, разработанные в статье, могут быть непосредственно применены в синтезе регулируемых параллельных манипуляционных механизмов с модульной структурой, предназначенных для приближенного воспроизведения заданных многоэтапных движений или множественных траекторий выходного звена.

Ключевые слова: соосные цилиндры, квадратическое приближение, алгебраическое отклонение, аппроксимационный синтез, реконфигурируемый манипуляционный механизм.