NPUA PROCEEDINGS, MECHANICS, MACHINE SIENCE, MACHINE-BUILDING. 2015. №2
UDC 621. 81/85

ON FATIGUE STRENGTH OF ANISOTROPIC MATERIALS

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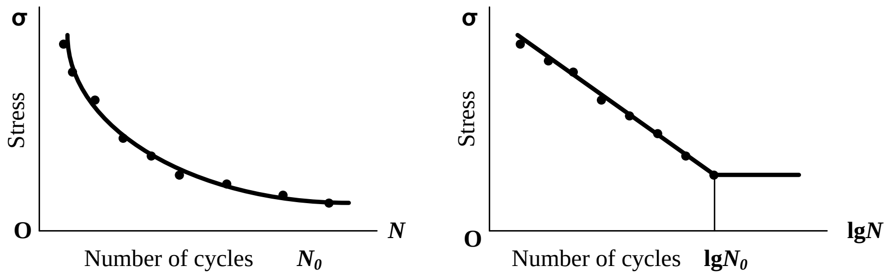
A study on the strength of materials in case of their cyclic loading within various ranges of alternating stresses is introduced. The graphical representation of relationship between the cycle average stress and the range of repeated or fluctuating stresses has been established. On the basis of a simplified limit stresses' graph for any asymmetric cycle, a stress calculation formula is derived. Considering the deformation process as an elasto-plastic phenomenon, a Mathie-Hill equation-type differential equation with variable constants has been obtained for a uniaxial oscillatory motion of a specimen. For a low frequency repeated cyclic load, an equation of a fatigue strength curve has been derived.

Keywords: fatigue strength, endurance limit, stress, deformation, cycle, oscillation.

Introduction. The purpose of this study is to analyze the endurance of materials under variable cyclic loading with various ranges of stress variation. The graphical relationship of the average cycle stress to the range of stresses has been obtained. From a simplified diagram of ultimate stresses, a computation formula was derived for the ultimate fatigue strength for any asymmetric cycle. Considering the deformation process as an elasto-plastic deformation, a differential equation of uniaxial oscillation of an object with a variable Mathieu-Hill's equation coefficient was set up. For a low frequency of stress alteration, an equation of the endurance graph was plotted. The loading pattern causing stresses of varying magnitude in cross-sections of machine and assembly parts is considered as the most typical. The failure of machine parts at such loads occurs at stresses below the ultimate strength and even yield point if only these changes of stresses are repeated sufficiently frequently.

The relationship between the number of cycles before the fracture occurs and the stresses causing the failure has been established on the basis of endurance curves (at least so far) plotted by experimentally obtained data σ, N in $\sigma - lg\,N$ or coordinates as shown in Fig.1.



a) b)

Fig. 1. Fatigue curves: a – in σ, N coordinates; b – in $\sigma, \lg N$ coordinates

These curves enable to define the most (maximum) stress of the cycle – the fatigue strength at which the sample specimen under the test does not fail at a (conditionally given) rather so called great number of cycles (stress variation). Most often, for ferrous metals it is enough to set up this stress on the basis of $N=10^7$ number cycles. At that, it is assumed that if the specimen does not fail when the base number of stress variation is reached while conducting tests, then the specimen will not fail at further tests.

A typical fracture due to fatigue has two zones of fatigue failure - fine-grained, almost smooth surface where fatigue crack penetrates deep into the cross-section, and the zone of static failure - coarse-grained textures over which final brittle fracture occurs. The pattern of fatigue failure zone depends on the number of loading cycles during which the crack develops since in the process of cyclic loading mutual rubbing and bearing of surfaces occur, which is followed by strain hardening.

The examination of strain hardening distribution on the fracture surface has shown that the most hardening takes place where the maximum number of contact cycles occurs in the crack initiation zones [1]. If the required life span of a part limited by a number of cycles is smaller than that of the base number then in computation it is necessary to make use of a limited fatigue point, which is the maximum value of the cycle stress that the given part can withstand. The endurance limit depends both on the nature of the cycle time-dependent variation – degree of the cycle asymmetry and the type of stressed state. In most cases fatigue tests are carried out for the symmetric cycle meanwhile in many cases, the computation of machine parts deals with stresses which change by asymmetric cycle [2]. A more precise idea on the actual strength of machine parts can be obtained from the results of a real-life test carried out on machines completely reproducing operational conditions of loading (according to the type of the stressed state, loading conditions, etc.). Weller was the first who systematically carried out experimental research to reveal and understand the phenomena of metal fatigue. He found that the stress range $R = \sigma_{max} - \sigma_{min}$, necessary to cause failure decreases as the average σ_m stress increases. Other researchers show that there is no general law coupling the average stress and the stress range [3].

Herber Baushinger [4] performed a series of fatigue experiments and on the basis of the obtained results proposed a parabolic law coupling the stress range *R* and the average stress σ_m. Fig. 2 shows this relation by parabolic curves where the average stress and the range of stress are expressed by the parts of the breaking limit. The stress range *R* turned to be the maximum in case of a symmetric cycle of stresses and tends to zero ($\sigma_m = 0$) when the average stress tends to the breaking point [5]. If the endurance limit for the symmetric cycle of stresses and the breaking point are known, the endurance limit for any asymmetric is known, then the breaking limit for any asymmetric cycle of stresses can be obtained from the limiting curves shown in Fig. 3.

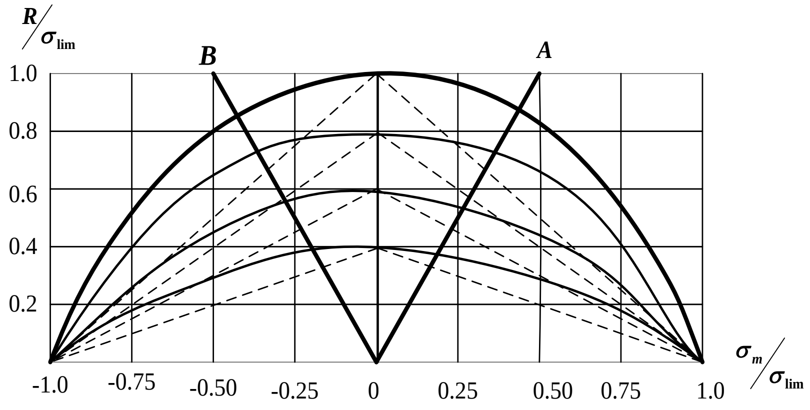


Fig. 2. The average stress of the cycle vs the stress range R in the parts of the breaking limit

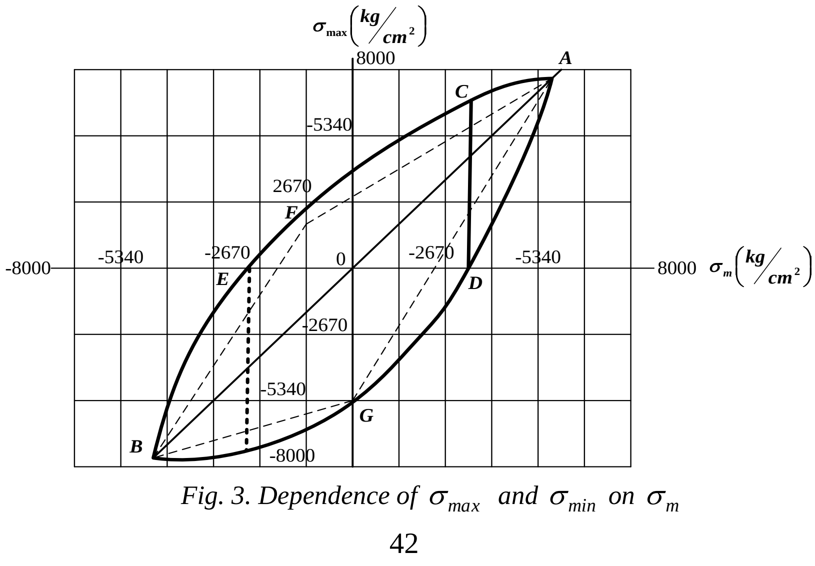


Fig. 3. Dependence of σ_{max} and σ_{min} on σ_m

The sloping upward *OA* and downward *OB* straight lines (Fig. 2) define the region *AOB* where the stress changes its sense in the cycle. The stress out of this region always remains tensile or compressive. The experimental determination of the values within the region *AOB* usually lies between the parabolas and the corresponding lines. If the stress is always tensile or compressive, the value of *R* is not only below the Herber parabola, and even lower of the corresponding lines [6].

Research method. Instead of representing the range of stresses as function σ_m, sometimes σ_{max} and σ_{min} are drawn in function σ_m (Fig. 4) and are computed from equation $\sigma_{max} = \sigma_m + R/2$, $\sigma_{min} = \sigma_m - R/2$ and plotted by adding $\pm$ *R*/2 to the ordinates of the AOB line sloping at an angle of 45^0.

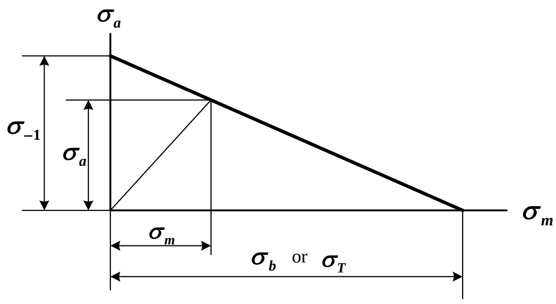


Fig. 4. Simplified diagram of ultimate stresses

The vertical line passing through point *D* corresponds to a tensile stress of the symmetric cycle.

Point A represents the ultimate tensile strength of a material and point B – the ultimate compression strength. The curves σ_{max} and σ_{min} represent the limiting states for fluctuating stresses. If the points corresponding to some real fluctuating stresses lie within *AEBDA*, the material will withstand that stress in an infinite number of cycles without failure. The curves σ_{max} and σ_{min} have been obtained from the parabolic curve (Fig. 2), however the parabolas in many cases are substituted by two sloping lines and the safety region shown in Fig. 3 is obtained in the form of the parallelogram *AFBCA*.

Many researchers find that the range of stresses depends not only on the magnitude but also the sense of the average stress σ_m. The ultimate compressive strength often differs from the ultimate tensile strength. For instance, for a number of materials the ultimate compressive strength (especially for brittle materials) is considerably more than the ultimate tensile strength. For composite materials, on the contrary, the ultimate compressive strength is almost 50 per cent lower than the ultimate tensile strength. Then, instead of symmetric parabolas (Fig. 2), we get asymmetric curves [7]. It is readily seen that such approaches of considering the asymmetry of cyclic stresses to define the ultimate strength in fluctuating stresses is rather complicated and the accuracy is arguable.

For a more simplified solution of the problem, it is assumed that the variation of the working load up to the ultimate state between the components of stress cycles σ_a and σ_{min} remains constant (σ_a / σ_{min} = *constant).*

Then the endurance limit for the given asymmetry coefficient *r* is written as:

$$\sigma_r = \sigma_a + \sigma_m, \quad r = \frac{\sigma_{min}}{\sigma_{max}}. \tag{1}$$

Fig. 4 shows a simplified diagram drawn by static strength (σ_b, σ_T) and endurance limit σ_{-1}.

According to the graph shown in Fig. 4 we have:

$$tg\beta = \frac{\sigma_a}{\sigma_m} = \frac{\sigma_{max} - \sigma_{min}}{\sigma_{max} + \sigma_{min}} = \frac{1-r}{1+r}. \tag{2}$$

From Eqs. (1) and (2) we have:

$$\sigma_r = \sigma_m + \sigma_a = \sigma_m \frac{2}{1+r}. \tag{3}$$

Equation of the AC line can be represented in the following form:

$$\frac{\sigma_a}{\sigma_{-1}} + \frac{\sigma_m}{\sigma_b} = 1. \tag{4}$$

Taking into account Eqs. (3) and (4) we get:

$$\sigma_r \geq \frac{2\sigma_{-1} \cdot \sigma_b}{\sigma_b(1-r) + (1+r)\sigma_{-1}}, \tag{5}$$

which means that it is sufficient to know the limit of static strength $\sigma_b(\sigma_T)$ and endurance limit σ_{-1} to determine the ultimate fatigue stress for any asymmetric cycle (Fig. 5).

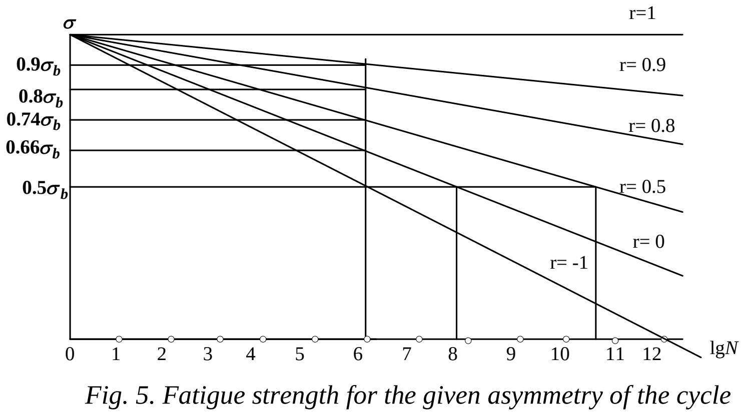


Fig. 5. Fatigue strength for the given asymmetry of the cycle

For evaluating the strength of machine parts and specimens, the value of endurance limits obtained by tests results needs to be corrected in view of a number of known strength factors (forms and absolute dimensions, state of a specimen surface, properties of surface layer, existing various concentrators changing patterns of loading etc.) [8].

Practically, it is very important to know how quickly the curve $\sigma - N$ tends to an asymptote since the number of cycles necessary for setting up the endurance limit characterizes the life-span of a machine part. Experiments show that for ferrous metals the endurance limit can be set up with adequate accuracy on the basis of $\sigma_b = 10$ millions cycles. For non-ferrous metals (for example, aluminum), there is no certain endurance limit, and ordinates of fatigue curve $\sigma - N$ decreases infinitely as the number of cycles rises. Determining the endurance limits for any material requires a great number of experiments. Therefore, the practical interest is in setting up a relationship between endurance limits and other mechanical properties which can be defined theoretically and by static tests. Although a great number of experimental data have been gathered it is still impossible to set up such a relationship [9]. In our opinion, it is exceedingly difficult, by summing up fatigue damages and microcracks, to implement the idea of the theoretically established functional relationship between the stress increases depending on the number of loading cycles. Expressions of integral sum of damages include initial magnitudes of microcracks and microflaws, which can be defined by correlation analysis of examination results of the test specimen. When it is considered that destruction most often occurs not where there is a large microcrack but in an absolutely other place and that the study of microcracks is not an easy work, then the significance of the present approach seems fantastic and ineffective, at times even meaningless. Therefore we consider the deformation process of a structure or a machine part not as pure elastic but as elasto-plastic. To describe an elasto-plastic deformation, a form of recording is used which frequently is applied in electrical engineering for defining the phase lag between voltage and current strength [10]. Then for the elasto-plastic deformation at uniaxial tension-compression, we get [11, 12]:

$$\sigma = E \cdot \varepsilon \cdot e^{a}, \tag{6}$$

where ε and σ are elastic deformation and stress, respectively, α is some constant which depends on the material property, the nature of deformation and the type of the stressed state, α at the same time is a small quantity (α <<1). Eq. (6) can be written in the following form [13]:

$$\sigma = E \cdot \varepsilon(1 + \alpha), \tag{7}$$

and since α <<1, then expanding e^{α} in Fourier series and neglecting small quantities, we get Eq. (7), where $\alpha \cdot \varepsilon$ is a residual plastic deformation after one stress change cycle. In multiple repeating of stress cycles, the plastic deformation will increase causing a change in the stiffness test specimen. If the initial tension or compression stiffness of the part or specimen under consideration were C_0, then after a certain number of stress alteration cycles the stiffness would decrease to the value $C_0 - C_k$, where

$$C_k = \alpha \cdot C_0 \cdot n . \tag{8}$$

Since the plastic deformation increases by exponential law $\left(\varepsilon_n = \varepsilon \cdot e^{\alpha}\right)$, then the specimen stiffness also decreases by exponential law, where n is the number of cycles in lg$N$ parts, that is $n = lg\,N$ or $n = 10^n$.

Then the stiffness of the test specimen, after load cyclic action, can be expressed as $C_o\left(1 - \alpha \cdot lg\,N\right)$. If the test specimen or a machine part is considered as an oscillation system with an exciting varying force $P(t) = P \cdot sin\,\omega t$, where ω is the angular frequency of oscillation $\left(\omega = 2\pi \cdot f,\quad f = H/t\right)$, then uniaxial oscillatory motion of the specimen can be described by a differential equation of varying coefficient [13]:

$$\ddot{y} + \frac{C_0}{m}\left(1 - \alpha \cdot lg\,N\right)y = \frac{P}{m} sin\,\omega t , \tag{9}$$

where y is the amplitude of oscillation, C_0 is the initial stiffness of the specimen, m is the unit volume mass of the specimen, P is the maximum load.

Research results. Eq. (9) is Mathieu-Hill's equation type with variable coefficient depending on time and the number of cycles. This equation has no analytic solution and can be solved by numerical methods only.

However, when the load is changed with frequency less than 20 *Hz*, inertia forces can be neglected, we get:

$$\frac{C_0}{m}\left(1 - \alpha \cdot lg\,N\right)y = \frac{P}{m} \cdot sin\,\omega t , \tag{10}$$

where $y = y_{max}\, sin\,\omega\, t$, then:

$$C_0\left(1 - \alpha \cdot lg\,N\right)y_{max} = P . \tag{11}$$

Assuming that $C_0 = \frac{P_{max}}{Y_{max}}$ where P_{max} is the maximum force which at the initial section F_0 of the specimen can cause a stress equal to σ_b or σ_T, σ_b, where σ_T is a pressure limit in static loading, we have:

$$P_{max}\left(1 - \alpha \cdot lg\,N\right) = P . \tag{12}$$

Dividing both sides by F_0, we have:

$$\sigma_b\left(1 - \alpha \cdot lg\,N\right) = \sigma , \tag{13}$$

or

$$\sigma + \alpha \cdot \sigma_b \cdot lg\, N = \sigma_b\,. \tag{14}$$

To determine coefficient α, the specimen is assumed to be a prismatic bar which measures α_0 in height, b_0 in width, and l_0 in initial length. Then the following can be written:

$$\int_{\alpha_0}^{\alpha} \frac{d\alpha}{\alpha} = -\mu \int_{l_0}^{l} \frac{dz}{z}; \quad \int_{b_0}^{b} \frac{db}{b} = -\mu \int_{l_0}^{l} \frac{dz}{z}\,; \tag{15}$$

$$\frac{\alpha}{\alpha_0} = \left(\frac{l_0}{l}\right)^{\mu}; \quad \frac{b}{b_0} = \left(\frac{l_0}{l}\right)\mu; \quad \frac{\alpha \cdot b}{\alpha_0 \cdot b_0} = \left(\frac{l_0}{l}\right)^{2\mu}; \quad \frac{F}{F_0} = \left(\frac{l_0}{l}\right)^{2\mu}, \tag{16}$$

where μ is the Poisson's ratio, F_0 is the initial cross-section area of the specimen, F is the cross-section area of the specimen after the first cycle of deformation.

The difference between the initial and final cross-sections areas is:

$$\Delta F = F_0 - F \quad \text{or} \quad \Delta F = F_0\left[1 - \left(\frac{l_0}{l}\right)^{2\mu}\right]. \tag{17}$$

Since (as follows from $\varepsilon = \varepsilon_y + \varepsilon_n = \varepsilon_y + \alpha \cdot \varepsilon_y$),

where

$$\varepsilon = \frac{\sigma_1}{E}; \quad \varepsilon_y = \frac{\sigma}{E}; \quad \sigma_1 = \frac{P}{F}; \quad \sigma = \frac{P}{F_0}\,. \tag{18}$$

From Eqs. (16), (17), and (18) we have:

$$\alpha = \left(\frac{l}{l_0}\right)^{2\mu} - 1\,. \tag{19}$$

Having in view that $l = l_0 + \Delta$ and $\frac{\Delta}{l_0} = \varepsilon_y = \frac{\sigma}{E}$, we get:

$$\alpha = \left(\frac{E + \sigma}{E}\right)^{2\mu} - 1\,. \tag{20}$$

Then Eq. (14) can be written in the following form:

$$\sigma + m_1\, lg\, N = \sigma_b\,, \tag{21}$$

where

$$m_1 = \sigma_b\left[\left(\frac{k \cdot E + \sigma_b}{k \cdot E}\right)^{2\mu} - 1\right], \quad k = \frac{\sigma_b}{\sigma}\,. \tag{22}$$

Conclusions. On the basis of the studies carried out and the experience of worldwide practice, for the first time, a formula is derived for calculating the ultimate stresses for any asymmetric cycle of stress alteration at known endurance limit in symmetric cycle. At a closer examination of the cyclic deformation process, and taking into account the plastic component differential equation for uniaxial oscillation of an object with variable coefficients of Mathieu-Hill–type and based on it, the equation of endurance curve has been derived where the mechanical characteristic of materials and the stress parameters have been taken into consideration.

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Received on 11.09.2015.
Accepted for publication on 15.12.2015.

ԱՆԻԶՈՏՐՈՊ ՆՅՈՒԹԵՐԻ ՀՈԳՆԱԾԱՅԻՆ ԱՄՐՈՒԹՅԱՆ ՀԱՐՑԻ ՄԱՍԻՆ

Հ.Գ. Շեկյան, Ն.Գ. Հովումյան

Դիտարկված է անիզոտրոպ նյութերի դիմադրողականությունը փոփոխական ցիկլիկ բեռնավորվածությունների դեպքում լարումների փոփոխության տարբեր միջակայքերի համար: Ստացված է ցիկլիկ լարման միջին արժեքի գրաֆիկական կախվածությունը լարման փոփոխման տիրույթից: Սահմանային լարումների պարզեցված դիագրամից ստացված է ցանկացած ասիմետրիկ ցիկլի համար հոգնածային հաշվարկային բանաձև: Դեֆորմացումը դիտարկելով որպես առաձգապլաստիկ երևույթ՝ ստացված է նմուշի միառանցք տատանումների Մաթե-Խիլի տիպի դիֆերենցիալ հավասարում, որը թույլ է տվել կառուցել հոգնածային կորի դիագրամը: Ցածր հաճախականությունների համար ստացված է նաև հոգնածային կորի հավասարումը:

Առանցքային բառեր. հոգնածային ամրություն, դիմացկունության սահման, լարում, դեֆորմացում, ցիկլ, տատանում:

К ВОПРОСУ ОБ УСТАЛОСТНОЙ ПРОЧНОСТИ АНИЗОТРОПНЫХ МАТЕРИАЛОВ

Г.Г. Шекян, Н.Г. Овумян

Рассмотрена выносливость материала при переменных циклических нагружениях с различными диапазонами изменения напряжений. Получена графическая зависимость среднего напряжения цикла от диапазона изменения напряжений. Из упрощенной диаграммы предельных напряжений получена формула расчета предела усталостной прочности для любого асимметрического цикла. Рассматривая деформирование как упругопластический процесс, получено дифференциальное уравнение одноосного колебания объекта с переменными коэффициентами типа Матье–Хилла. Для малых частот изменения напряжений получено уравнение кривой выносливости.

Ключевые слова: усталостная прочность, предел выносливости, напряжение, деформация, цикл, колебание.